A Unique Maximally Doubled Minimal Dominating Set in a Triangulated Ladder

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Abstract

Consider a $2 \times n$ ladder with a consistently oriented diagonal in each cell. When $n = 3m$, there is exactly one minimal dominating set with m columns containing both vertices: it selects both vertices in every third column, beginning with column 2. We prove this using private neighbors and report the scope of computational and Lean checks.

1. Introduction

Minimal dominating sets can be refined by both their size and their number of *doubly occupied* columns. The result here identifies a unique extremal configuration for the consistently triangulated ladder. Although boundary-state recurrences and half-turn parity congruences motivate the broader topic, neither is established here.

2. Notation

The ladder has vertices T_i, B_i for $1 \le i \le n$. Its edges are T_iB_i for $1 \le i \le n$, and T_iT_{i+1}, B_iB_{i+1}, and T_iB_{i+1} for $1 \le i < n$. A set is *dominating* if every vertex belongs to it or has a neighbor in it; it is *minimal dominating* if deleting any selected vertex destroys domination. Column i is *doubly occupied* when both T_i and B_i are selected. Write $N[x]$ for the closed neighborhood of x.

3. Results

The following proof is reproduced verbatim from the verified proof.

Theorem. *Let $n = 3m$, where $m \ge 1$. The unique minimal dominating set of the ladder with exactly m doubly occupied columns is*

$$D_0 = \{T_{3j+2}, B_{3j+2} : 0 \le j < m\}.$$

In particular, it has size $2m$.

Lemma 1 (Private neighbors)**.** *A dominating set D is minimal if and only if, for every $v \in D$, there is a vertex x whose closed neighborhood satisfies*

$$N[x] \cap D = \{v\}.$$

Proof. If D is minimal, then $D \setminus \{v\}$ fails to dominate some vertex x. Since D dominates x, the only member of D in $N[x]$ is v. Conversely, if such an x exists for every $v \in D$, deleting v leaves x undominated. Thus no vertex can be deleted while preserving domination.

□

Lemma 2 (A doubly occupied column is interior)**.** *If column i is doubly occupied in a minimal dominating set D, then $1 < i < n$. Moreover,*

$$N[T_{i+1}] \cap D = \{T_i\}, \qquad N[B_{i-1}] \cap D = \{B_i\}.$$

Proof. Both T_i and B_i belong to D. From the specified edges, the only vertex of $N[T_i]$ that is not also in $N[B_i]$ is T_{i+1}, when that vertex exists. By Lemma 1, T_i needs a private neighbor. Hence $i < n$, and that private neighbor must be T_{i+1}.

Likewise, the only vertex of $N[B_i]$ that is not also in $N[T_i]$ is B_{i-1}, when it exists. Lemma 1 therefore gives $i > 1$ and the asserted equality for B_{i-1}.

□

Lemma 3 (Doubly occupied columns are at least three columns apart)**.** *If $i < k$ are doubly occupied columns of a minimal dominating set, then $k - i \geq 3$.*

Proof. By Lemma 2, T_{i+1} is private to T_i. If $k = i + 1$, then T_{i+1} itself belongs to D, contradicting that privacy. If $k = i + 2$, then $T_{i+2} \in D$ is adjacent to T_{i+1}, giving the same contradiction.

□

Lemma 4 (The positions are forced)**.** *If $n = 3m$ and a minimal dominating set has exactly m doubly occupied columns, their indices are $2, 5, \ldots, 3m - 1$.*

Proof. Write their indices as $i_1 < \cdots < i_m$. Lemmas 2 and 3 give

$$2 \leq i_1, \qquad i_{r+1} \geq i_r + 3, \qquad i_m \leq 3m - 1.$$

Consequently,

$$3m - 1 \geq i_m \geq i_1 + 3(m - 1) \geq 3m - 1.$$

Equality holds throughout, so $i_1 = 2$ and every successive difference is 3.

□

Lemma 5. *D_0 is minimal dominating.*

Proof. Every vertex in a selected column belongs to D_0. In column $3j + 1$, the top and bottom vertices are adjacent to T_{3j+2} and B_{3j+2}, respectively. In column $3j + 3$, the top and bottom vertices are adjacent to T_{3j+2} and B_{3j+2}, respectively. Thus D_0 dominates every vertex.

For each j, the only member of D_0 in $N[T_{3j+3}]$ is T_{3j+2}: the other selected columns are three columns away from column $3j + 2$. Similarly,

$$N[B_{3j+1}] \cap D_0 = \{B_{3j+2}\}.$$

Every member of D_0 therefore has a private neighbor. Lemma 1 shows that D_0 is minimal.

□

Proof of the theorem. Let D be a minimal dominating set with exactly m doubly occupied columns. By Lemma 4, those columns are $2, 5, \ldots, 3m - 1$, so $D_0 \subseteq D$. Lemma 5 shows that D_0 already dominates the ladder. If D contained any additional vertex, deleting that vertex would leave the dominating set D_0 within D, contrary to minimality. Hence $D = D_0$. Lemma 5 also establishes its existence, and its m doubly occupied columns give $|D_0| = 2m$.

□

4. Computational verification

Subsets were tested for domination, minimality, and exactly m doubly occupied columns for $m = 1, 2, 3, 4$, corresponding to $n = 3, 6, 9, 12$. The reported run tested **3,310,173 candidate sets** and found no counterexample. An independent check reported PASS for each lemma and the theorem.

Lean 4 status for the theorem: compiles with `sorry`, not machine-checked sorry-free. The file compiled but reported five `sorry` warnings; `main_theorem` depends on `sorryAx`. Its back-translation was judged **FAITHFUL**: the Lean statement uses zero-based columns $3j + 1$, corresponding to the paper's one-based columns $3j + 2$. A sorry-free Lean proof would certify its Lean statement, which would still have to be read against the informal theorem; that standard has not been reached here.

5. Limitations and open questions

The theorem supplies one extremal term, not a boundary-state recurrence or a half-turn parity congruence. Deriving either for the full enumeration by set size and doubly occupied columns remains open here. No `## GAP` was recorded in the supplied proof; the unresolved formalization obligations are the five Lean `sorry` placeholders.

Novelty

The exact recorded search queries were:

1. `"triangular ladder graph" "minimal dominating set" "every third rung" uniqueness`
2. `triangulated ladder graphs domination minimal dominating sets upper domination number`
3. `ladder graph minimal dominating sets private neighbors column-state transfer matrix`

The record lists **28 retrieved hits** in total—15 from arXiv and 13 from Crossref—and no matching hit. OpenAlex attempts for all three queries returned HTTP 429 errors, so they supplied **zero hits**. The three additional strings marked as *follow-up queries* in the record are not documented as having been run.

This is a **bounded automated search over those indexes**, not a literature review. It is blind to books, journals outside them, and results phrased differently; OpenAlex was additionally unavailable for these queries. The evidence base is recorded as *retrieval*, not *memory-only*, but the search does not establish novelty.

Models

Unique maximally doubled minimal dominating set (`verified`)

- Proposed the claim: gpt-6-sol
- Counterexample search: gpt-6-sol
- Wrote the proof: gpt-6-sol
- Referee: gpt-6-sol
- Independent check script: gpt-6-sol
- Lean formalization: gpt-6-sol
- Faithfulness judge: gpt-6-sol