A parity identity for maximal matchings of open and cyclic ladders

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Abstract

Let $L_n(x)$ and $C_n(x)$ count maximal matchings by size in the open and cyclic ladders, respectively. Row exchange pairs all nonfixed maximal matchings, yielding the coefficientwise identity

$$C_n(x) \equiv L_n(x) + x^2 L_{n-2}(x) \pmod 2 \qquad (n \geq 3).$$

The proof reduces the surviving terms to fixed perfect matchings and separates those using the closing edges.

1. Introduction

Closing an open ladder adds two horizontal edges between its end columns. Counting maximal matchings after this change is sensitive to whether end vertices are unmatched. Here, modulo two, row-exchange symmetry removes that complication: every fixed maximal matching is perfect. The resulting open-to-cyclic correction is a shifted polynomial for a shorter ladder.

2. Notation

The open ladder is $P_n \square K_2$, and the cyclic ladder is $C_n \square K_2$. For either graph G, its *maximal-matching polynomial* is

$$\sum_{\substack{M \text{ a maximal} \\ \text{matching of } G}} x^{|M|}.$$

Write $L_n(x)$ and $C_n(x)$ for these polynomials for the open and cyclic ladders. Maximal means maximal under inclusion, not necessarily of maximum size.

3. Results

Theorem. *For every integer $n \geq 3$, coefficient by coefficient,*

$$C_n(x) \equiv L_n(x) + x^2 L_{n-2}(x) \pmod 2.$$

Proof.

Let σ be the automorphism that exchanges the two rows of a ladder, and write $F_G(x)$ for the generating polynomial of the σ-fixed maximal matchings of a ladder G.

Lemma 1. *For either ladder G, its maximal-matching polynomial is congruent to $F_G(x)$ modulo 2.*

Proof. The map $M \mapsto \sigma(M)$ preserves both maximality and the number of edges. Every matching not fixed by σ belongs to a two-element orbit $\{M, \sigma(M)\}$. The two members contribute $2x^{|M|}$, which is zero modulo 2. Thus only fixed matchings contribute modulo 2.

□

Lemma 2. *A σ-fixed matching of either ladder is maximal if and only if it is perfect.*

Proof. Every perfect matching is maximal. Conversely, suppose a fixed matching M leaves a vertex v unmatched. Since $M = \sigma(M)$, its row-mate $\sigma(v)$ is also unmatched. The rung joining these vertices can then be added to M, contradicting maximality.

□

A fixed perfect matching consists of rungs and pairs of horizontal edges: each rung is fixed by σ, while each horizontal edge is exchanged with the corresponding edge in the other row.

Lemma 3. *The fixed perfect matchings of the cyclic ladder on $n \geq 3$ columns that do not use the horizontal-edge pair between columns n and 1 are in weight-preserving bijection with the fixed perfect matchings of the open ladder on n columns.*

Proof. Removing the two edges between columns n and 1 leaves the open ladder. A matching that uses neither edge remains the same perfect matching after their removal, and its fixedness and number of edges are unchanged. Conversely, any fixed perfect matching of the open ladder is a fixed perfect matching of the cyclic ladder using neither added edge.

□

Lemma 4. *The fixed perfect matchings of the cyclic ladder on $n \geq 3$ columns that use the horizontal-edge pair between columns n and 1 have generating polynomial $x^2 F_{P_{n-2}\square K_2}(x)$.*

Proof. That pair matches all four vertices in columns n and 1. Removing it leaves a fixed perfect matching on columns $2, \ldots, n-1$, which form an open ladder on $n-2$ columns. Conversely, adjoining the pair to any such matching gives a fixed perfect matching of the cyclic ladder. The adjoining operation adds exactly two edges, so it multiplies each weight by x^2.

□

By Lemmas 3 and 4, the fixed perfect matchings of the cyclic ladder have generating polynomial

$$F_{C_n\square K_2}(x) = F_{P_n\square K_2}(x) + x^2 F_{P_{n-2}\square K_2}(x).$$

Lemma 2 identifies these as the fixed maximal matchings. Applying Lemma 1 to all three ladders therefore gives, coefficient by coefficient,

$$C_n(x) \equiv L_n(x) + x^2 L_{n-2}(x) \pmod 2 \qquad (n \geq 3).$$

□

4. Computational verification

All matchings of both ladders were enumerated for $3 \leq n \leq 10$, filtered for inclusion-maximality, and compared at every coefficient x^k, $0 \leq k \leq n$. No counterexample was found in the 60 reported (n, k) cases. A separate check reported Lemmas 1–2 on open ladders through $n = 8$ and cyclic ladders for $3 \leq n \leq 8$, and Lemmas 3–4 and the theorem for $3 \leq n \leq 8$, all passing.

Lean 4 status: failed to compile. The attempted file reports an unsolved goal and six `sorry` warnings; its axiom report includes `sorryAx`. It is neither a compiling proof with `sorry` nor a machine-checked sorry-free proof. **Back-translation faithfulness verdict: undetermined**, because the Lean theorem statement itself was not supplied for comparison with the informal statement.

5. Limitations and open questions

The identity is a parity result; it does not give an integer-coefficient end-extension recurrence refined by matching size and the unmatched status of both end vertices. Finding such a recurrence remains open here.

GAP: formal verification

The attempted Lean development did not compile and contains admitted steps. Its lemmas and final theorem therefore have no claimed Lean certification. A sorry-free Lean proof would certify its *Lean statement*, which would still need to be read against the informal theorem; that comparison cannot be completed from the supplied record.

Novelty

The exact recorded search queries were:

1. `"maximal matching polynomial" ladder prism "mod 2" "x^2"`
2. `"maximal matching generating polynomial" open ladder cyclic ladder prism graphs parity congruence`
3. `transfer matrix GF(2) maximal matchings ladder graphs path cycle boundary states`

The recorded returns comprise 15 arXiv, 15 Crossref, and 5 OpenAlex entries; none was identified as a matching result. These counts are from a **bounded automated search over those indexes**, not a literature review. The search is blind to books, journals outside them, and work phrased differently. Novelty should therefore be regarded as tentative.