Triangle-Orbit Congruences for a Putative Conway 99-Graph: Order-Three and Involution Automorphisms

Prathik Katukala*

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Abstract

We study necessary conditions on a putative strongly regular graph with parameters $(99, 14, 1, 2)$ — a Conway 99-graph, the smallest undecided case of the form $(v, k, 1, 2)$. We prove an elementary congruence: for an automorphism of prime order p of a finite graph, the number of triangles is congruent modulo p to the number of triangles preserved setwise. Combining this with a spectral projector for the 3-eigenspace, we obtain necessary conditions in the surviving automorphism cases: (i) a fixed-point-free order-3 automorphism has triangle-orbit parameter $T \in \{6, 27\}$; (ii) an order-3 automorphism fixing a triangle has $t \equiv 2 \pmod{21}$ and, since every triangle-orbit is a free orbit, $t = 2$; (iii) an involution fixing exactly one vertex has $e \equiv 0 \pmod{7}$ with e odd, so $e \in \{7, 21, 35, 49\}$, conditional on a cited fixed-point classification. We also prove the first of two quotient identities, $\sum_{P<Q} f_{PQ} = 80 - t$; the second, $\sum_{P<Q} f_{PQ}^2 = 108 - 3t$, is not established and is not used. These are necessary conditions only: because a putative graph could be asymmetric, excluding prime-order automorphisms cannot settle existence. The order-3 fixed-triangle case is known to contain no graph, so the reduction there is an internal parameter restriction of a closed case. Novelty of the refinements is not established.

Keywords: strongly regular graph; Conway 99-graph; automorphism; triangle orbit; spectral projector; orbit congruence; quotient matrix.

1 Introduction

A strongly regular graph with parameters (v, k, λ, μ) is a k-regular graph on v vertices in which every pair of adjacent vertices has exactly λ common neighbours and every pair of non-adjacent vertices has exactly μ common neighbours. The parameter set $(99, 14, 1, 2)$ is the smallest of the form $(v, k, 1, 2)$ whose existence is undecided; it is the subject of Conway's \$1000 problem [4] (Problem 2 of that list). Throughout, Γ denotes a putative such graph, and A its adjacency matrix.

The automorphism group $G = \mathrm{Aut}(\Gamma)$ is heavily constrained. The only possible prime divisors of $|G|$ are $2, 3, 7, 11$ [6]; the order-11 case does not occur [7]; and $7 \mid |G|$ forces $G \cong \mathbb{Z}_7$ [3]. The order-7 case (one fixed vertex) and the order-3 case with a fixed triangle have been eliminated by orbit-matrix enumeration [1, 2]. Thus, among non-trivial automorphisms of prime order, the cases that remain open are order 3 with no fixed vertex (left unfinished in [1, 2]) and order 2 (not run there). This paper gives congruences for both, as well as an internal parameter restriction ($t = 2$) for the already-excluded order-3 fixed-triangle case. Because Γ could be asymmetric ($|\mathrm{Aut}(\Gamma)| = 1$), restricting or eliminating prime-order automorphisms constrains only symmetric candidates and cannot by itself settle existence.

*ORCID: 0009-0003-9585-8738.

Research question and method. The question is whether any graph with parameters $(99, 14, 1, 2)$ exists; this paper does not answer it. It constrains the automorphism cases that remain after the published exclusions, by the following pipeline: (1) fix the triangle count 231; (2) let a prime-order automorphism σ act on the triangle set, so that the number of σ-invariant triangles determines the total modulo p (the triangle-orbit congruence, Theorem 3.1); (3) identify the invariant triangles in each case (Lemma 3.3); (4) add an independent congruence from the spectral projector; (5) intersect the congruences to restrict the triangle-orbit parameter; (6) use computation only to check exact algebra and small examples, never as a substitute for proof. The results are reductions, not a resolution.

Contribution categories.

- *Proved here:* the triangle-orbit congruence (Theorem 3.1), the fixed-vertex lemma (Lemma 5.1), the order-3 congruences (Theorems 4.1 and 5.2), the involution congruence (Theorem 6.2), the projector identities (Lemma 2.4), the quotient/equitable-partition identity (Lemma 2.7), the orbit-type and quotient-entry bound (Lemma 5.3), the sharpened triangle count $t = 2$ (Corollary 5.4), and the first quotient identity $\sum_{P<Q} f_{PQ} = 80 - t$ (Proposition 5.5).
- *Imported (external):* the involution fixed-point classification (Theorem 6.1), from [6]; used only as an input, and not independently verified here.
- *Not established:* the second quotient identity $\sum_{P<Q} f_{PQ}^2 = 108 - 3t$ (Remark 5.6); it is neither proved nor used. No result of this paper depends on it.
- *Reproduced:* the classical congruence $T \equiv 6 \pmod 7$ [6], re-derived by the trace method.
- *Computational:* exact checks of the projector algebra and the quotient identity on Paley(9) (Appendix A).
- *Unresolved:* existence of Γ; the second quotient identity; the novelty of the case reductions (§8).

2 Preliminaries

Definition 2.1. Γ is a finite simple undirected graph; A is its adjacency matrix; $\mathrm{Aut}(\Gamma)$ is its automorphism group. For $\sigma \in \mathrm{Aut}(\Gamma)$ put $\alpha_0(\sigma) := |\mathrm{Fix}(\sigma)| = \mathrm{tr}(\sigma)$ and $\alpha_1(\sigma) := \#\{v : v \sim \sigma(v)\} = \mathrm{tr}(\sigma A)$. A triangle is a 3-clique, and it is σ-invariant if σ maps it to itself as a set. A *triangle-orbit* is a σ-orbit of vertices whose three vertices induce a triangle (equivalently, an orbit of internal degree 2). For $|\sigma| = 3$, T denotes the number of triangle-orbits in the fixed-point-free case and t the number in the three-fixed-vertex case. Each triangle-orbit yields exactly one σ-invariant triangle and contributes 3 to $\alpha_1(\sigma)$, so $\alpha_1 = 3T$ (resp. $\alpha_1 = 3t$).

For an involution τ with $\mathrm{Fix}(\tau) = \{z\}$: e is the number of *edge 2-orbits*, i.e. orbits $\{x, \tau x\}$ whose two vertices are adjacent (equivalently, the number of edges fixed by τ); n is the number of τ-invariant triangles, and $\alpha_0(\tau) = 1$, $\alpha_1(\tau) = 2e$. (The phrase "edge 2-orbits" always means these fixed edges, not all orbits of the induced action on the edge set.)

Lemma 2.2. *If Γ has parameters $(99, 14, 1, 2)$ then $A^2 = 12I - A + 2J$.*

Proof. For a strongly regular graph, $(A^2)_{xy} = k\delta_{xy} + \lambda A_{xy} + \mu(1 - \delta_{xy} - A_{xy})$, giving $A^2 = (k - \mu)I + (\lambda - \mu)A + \mu J = 12I - A + 2J$. □

Lemma 2.3. $\mathrm{Spec}(A) = \{14^1, 3^{54}, (-4)^{44}\}$.

Proof. On the subspace orthogonal to the all-ones vector, an eigenvector of A with eigenvalue θ satisfies $\theta^2+\theta-12=0$, so $\theta \in \{3,-4\}$. The all-ones vector has eigenvalue $k=14$. The multiplicities satisfy $1+f_3+f_{-4}=99$ and $14+3f_3-4f_{-4}=0$ (trace), giving $f_3=54$, $f_{-4}=44$. □

Lemma 2.4 (spectral projector)**.** *The projector onto the* 3*-eigenspace is*

$$E_3 = \frac{(A-14I)(A+4I)}{(3-14)(3+4)} = \frac{A^2-10A-56I}{-77} = \frac{44I+11A-2J}{77},$$

where the last equality uses Lemma 2.2.

Proof. Direct expansion of $\prod_{\theta\neq 3}(A-\theta I)/(3-\theta)$ using Lemma 2.2. Checked by an exact-arithmetic script (Appendix A). □

Remark 2.5. Since $\sigma A = A\sigma$, the automorphism σ preserves the 54-dimensional real 3-eigenspace $V_3 = \mathrm{im}(E_3)$, so $\sigma|_{V_3}$ is an orthogonal operator of prime order p. Since $\sigma J = J$ for any permutation matrix σ, we have $\mathrm{tr}(\sigma J) = 99$, and

$$\mathrm{tr}(\sigma E_3) = \frac{44\alpha_0(\sigma) + 11\alpha_1(\sigma) - 2\,\mathrm{tr}(\sigma J)}{77} = \frac{4\alpha_0(\sigma)+\alpha_1(\sigma)-18}{7}.$$

Lemma 2.6. *If $\lambda = 1$ then $N_\triangle$, the number of triangles, equals* 231*.*

Proof. Every edge lies in exactly one triangle, so $3N_\triangle = |E| = 99\cdot 14/2 = 693$, hence $N_\triangle = 231$. □

Lemma 2.7 (quotient identity)**.** *Let σ be a permutation automorphism of a graph Γ with parameters (v,k,λ,μ), let $O_1,\dots,O_b$ be its vertex orbits, and let M be the quotient matrix $M_{ij} = |N(x_i)\cap O_j|$ for a representative $x_i \in O_i$. Then M is well defined and*

$$(M^2)_{ij} = (k-\mu)\,\delta_{ij} + (\lambda-\mu)\,M_{ij} + \mu\,|O_j|.$$

For $(99,14,1,2)$ this reads $(M^2)_{ij} = 12\delta_{ij} - M_{ij} + 2|O_j|$.

Proof. The orbits of any automorphism form an equitable partition: $|N(x)\cap O_j|$ is constant on O_i, because σ carries $N(x)$ to $N(\sigma x)$ and fixes O_j setwise. Writing $u_i = \sum_{x\in O_i} e_x$ for the orbit indicator and using $\sum_i u_i = \mathbf{1}$, one computes $Au_i = \sum_j M_{ji}u_j$ and hence $A^2u_i = \sum_j (M^2)_{ji}u_j$. Since $A^2 = (k-\mu)I + (\lambda-\mu)A + \mu J$ and $Ju_i = |O_i|\,\mathbf{1} = |O_i|\sum_j u_j$, comparison of the coefficient of u_j gives $(M^2)_{ji} = (k-\mu)\delta_{ij} + (\lambda-\mu)M_{ji} + \mu|O_i|$; transposing yields the stated identity. □

3 The triangle-orbit congruence

Theorem 3.1. *Let Γ be a finite graph and let $\sigma \in \mathrm{Aut}(\Gamma)$ have prime order p. With $N_\triangle$ the total number of triangles and N_σ the number preserved setwise by σ,* $\quad N_\triangle \equiv N_\sigma \pmod p$.

Proof. The cyclic group $\langle\sigma\rangle$ acts on the set of triangles. Each orbit has size 1 or p (orbit–stabiliser). The size-1 orbits are exactly the setwise-invariant triangles. Hence $N_\triangle = N_\sigma + pm$ for some integer $m \geq 0$. □

Remark 3.2. Local linearity (every edge in exactly one triangle) is not needed for Theorem 3.1; it enters only through the value $N_\triangle = 231$ (Lemma 2.6).

Lemma 3.3. *For $p=3$, a σ-invariant triangle is either entirely fixed or is a single vertex-orbit.*

Proof. σ induces on the three vertices a permutation of order dividing 3: the identity (all fixed) or a 3-cycle (one orbit). □

4 Order-three automorphisms without fixed vertices

Let $|\sigma| = 3$ with $\alpha_0 = 0$; the 99 vertices form 33 orbits of size 3.

Theorem 4.1. *For a putative Γ with a fixed-point-free order-3 automorphism, the number T of triangle-orbits satisfies $T \equiv 0 \pmod 3$ and $T \equiv 6 \pmod 7$, hence $T \in \{6, 27\}$.*

Proof. Mod 3. By Lemma 3.3 the invariant triangles are the triangle-orbits (there are no fixed vertices), so $N_\sigma = T$. By Theorem 3.1 and $N_\triangle = 231$, $T \equiv 231 \equiv 0 \pmod 3$.

Mod 7. Each triangle-orbit contributes 3 to α_1, so $\alpha_1 = 3T$. With $\alpha_0 = 0$ and $\mathrm{tr}(\sigma J) = 99$, $\mathrm{tr}(\sigma E_3) = \frac{11 \cdot 3T - 2 \cdot 99}{77} = \frac{3(T-6)}{7}$. On the 3-eigenspace, σ has eigenvalues $1, \omega, \omega^2$ with multiplicities m_0, m_1, m_1, $m_0 + 2m_1 = 54$; hence $\mathrm{tr}(\sigma E_3) = m_0 - m_1 = (3m_0 - 54)/2$. Equating, $m_0 = \frac{2T+114}{7} \in \mathbb{Z} \iff T \equiv 6 \pmod 7$.

Finally $T \le 33$ (there are 33 orbits); the values $\equiv 6 \pmod 7$ are $\{6, 13, 20, 27\}$, and imposing $T \equiv 0 \pmod 3$ leaves $\{6, 27\}$. □

5 Order-three automorphisms with three fixed vertices

Let $|\sigma| = 3$ with $\alpha_0 = 3$; there are 32 orbits of size 3. By the classification of [6] the fixed set is a triangle; this is also forced by the degree count below.

Lemma 5.1. *The three fixed vertices form a triangle K_3.*

Proof. For a fixed vertex s, the neighbourhood is σ-invariant, so each 3-orbit is wholly adjacent or wholly non-adjacent to s. Hence $\deg(s) = \deg_{\mathrm{Fix}}(s) + 3 \cdot (\#\text{adjacent orbits}) = 14$, so $\deg_{\mathrm{Fix}}(s) \equiv 14 \equiv 2 \pmod 3$. Since there are only two other fixed vertices, $\deg_{\mathrm{Fix}}(s) \le 2$, so $\deg_{\mathrm{Fix}}(s) = 2$ for each s: the fixed set is K_3. □

Theorem 5.2. *For a putative Γ with an order-3 automorphism fixing exactly three vertices, the number t of triangle-orbits satisfies $t \equiv 2 \pmod 3$ and $t \equiv 2 \pmod 7$, hence $t \equiv 2 \pmod{21}$, i.e. $t \in \{2, 23\}$.*

Proof. Mod 3. By Lemma 5.1 the fixed set is K_3, one triangle; by Lemma 3.3 the invariant triangles are that K_3 and the triangle-orbits, so $N_\sigma = 1 + t$. With $N_\triangle = 231$, $231 = 1 + t + 3m$, i.e. $t \equiv 2 \pmod 3$.

Mod 7. Here $\alpha_0 = 3$ and $\alpha_1 = 3t$, so $\mathrm{tr}(\sigma E_3) = \frac{44 \cdot 3 + 11 \cdot 3t - 2 \cdot 99}{77} = \frac{3(t-2)}{7} = \frac{3m_0 - 54}{2}$, giving $m_0 = (2t + 122)/7 \in \mathbb{Z}$, i.e. $t \equiv 2 \pmod 7$. Combined, $t \equiv 2 \pmod{21}$; with $t \le 32$ this gives $t \in \{2, 23\}$, sharpened to $t = 2$ in Corollary 5.4. □

We now prove the first quotient identity. Throughout, let $F = \{z_1, z_2, z_3\} = \mathrm{Fix}(\sigma)$ be the fixed triangle; there are $(99 - 3)/3 = 32$ vertex 3-orbits. Since two fixed vertices have a unique common neighbour (the third), no orbit is adjacent to two fixed vertices; since each fixed vertex has $14 - 2 = 12$ neighbours outside F, exactly 4 orbits meet each z_i, and these $3 \cdot 4 = 12$ orbits are distinct. Call these 12 orbits the *type-1* orbits (write T_1 for their set) and the remaining $32 - 12 = 20$ orbits the *free* orbits F_0 (here "free" means "with no edge to $\mathrm{Fix}(\sigma)$", *not* the group-theoretic sense of trivial stabiliser, which every size-3 orbit already has). For free orbits P, Q write $f_{PQ} := M_{PQ}$.

Lemma 5.3 (orbit types and quotient-entry bound)**.** (a) *The subgraph induced on any σ-orbit is empty or a triangle K_3; hence $M_{PP} \in \{0, 2\}$ for every size-3 orbit P, and $M_{PQ} \le 2$ for distinct size-3 orbits P, Q.* (b) *No orbit adjacent to a fixed vertex is a triangle; consequently every triangle-orbit is a free orbit.*

Proof. (a) The induced subgraph on an orbit is σ-invariant; applying σ to any edge of it produces all three edges of the orbit, so the induced graph is empty or K_3. In particular $M_{PP} \in \{0, 2\}$. For distinct size-3 orbits P, Q, the entry $M_{PQ} = |N(x) \cap Q|$ is well defined and constant on P. Suppose $M_{PQ} = 3$. Then every vertex of P is adjacent to all three vertices of Q; taking two distinct vertices $x, x' \in P$, both are adjacent to all of Q, so $Q \subseteq N(x) \cap N(x')$ and $|N(x) \cap N(x')| \geq 3$. But two distinct vertices of a strongly regular graph have $\lambda = 1$ or $\mu = 2$ common neighbours, at most 2 — a contradiction. Hence $M_{PQ} \leq 2$.

(b) Let Q be an orbit adjacent to the fixed vertex z, so $M_{zQ} = |N(z) \cap Q| = 3$ (all three vertices of Q lie in $N(z)$). Apply Lemma 2.7 at (z, Q):

$$(M^2)_{zQ} = 12 \cdot 0 - M_{zQ} + 2 \cdot 3 = 3.$$

Expanding the left side, $\sum_R M_{zR} M_{RQ} = \underbrace{\sum_{R \text{ size-3}} 3\,[R \sim z]\, M_{RQ}}_{3 \sum_{R \sim z} M_{RQ}} + \underbrace{\sum_{z' \neq z} M_{zz'} M_{z'Q}}$. Here $M_{zz'} = 1$ and $M_{z'Q} = |N(z') \cap Q| = 3[Q \sim z']$. Since Q is adjacent to the fixed vertex z, it is adjacent to no other fixed vertex, so $[Q \sim z'] = 0$ and the second sum is 0. Hence $3 \sum_{R \sim z} M_{RQ} = 3$, i.e. $\sum_{R \sim z} M_{RQ} = 1$. The orbit Q itself is one of the four orbits $R \sim z$, so $M_{QQ} + \sum_{R \sim z,\, R \neq Q} M_{RQ} = 1$; all terms are non-negative, so $M_{QQ} \leq 1$, and since $M_{QQ} \in \{0, 2\}$ we get $M_{QQ} = 0$. Thus Q is independent. Every orbit adjacent to a fixed vertex is type-1, so no type-1 orbit is a triangle; by Lemma 3.3 the only other invariant triangle is the fixed one, so every triangle-orbit is free. □

Corollary 5.4. *For a putative* Γ *with an order-*3 *automorphism fixing a triangle,* $t = 2$.

Proof. By Lemma 5.3(b) every triangle-orbit is a free orbit, and there are exactly 20 free orbits; hence $t \leq 20$. By Theorem 5.2, $t \equiv 2 \pmod{21}$. The only integer congruent to 2 modulo 21 and at most 20 is 2; hence $t = 2$. □

Proposition 5.5 (first quotient identity)**.** *For a putative* Γ *with an order-*3 *automorphism fixing a triangle,*

$$\sum_{P < Q \text{ free}} f_{PQ} = 80 - t.$$

Proof. For a free orbit P and a fixed vertex z, apply Lemma 2.7 at (P, z): $(M^2)_{Pz} = 12 \cdot 0 - M_{Pz} + 2 \cdot 1 = 2$, and the left side is $\sum_{R \sim z} M_{PR} = 2$. Summing over the three fixed vertices, $\sum_{R \in T_1} M_{PR} = 6$ for every free orbit P. The row sum of M at P is the degree 14; with no neighbours in F this reads $a_P + b_P + 6 = 14$, where $a_P := M_{PP} \in \{0, 2\}$ and $b_P := \sum_{Q \neq P,\ Q \text{ free}} M_{PQ}$. Hence $a_P + b_P = 8$ for every free P.

Summing over the 20 free orbits, $\sum_{P \in F_0} a_P + \sum_{P \in F_0} b_P = 160$. By Lemma 5.3, $\sum_{P \in F_0} a_P = 2t$. And each free–free edge between distinct orbits is counted once in each direction, so $\sum_{P \in F_0} b_P = 2 \sum_{P < Q \text{ free}} f_{PQ}$. Thus $2t + 2 \sum_{P < Q} f_{PQ} = 160$, i.e. $\sum_{P < Q} f_{PQ} = 80 - t$. □

Remark 5.6 (the second quotient identity is not established). The same quotient identity yields a companion formula. Applying Lemma 2.7 at (P, P) for a free orbit P gives $(M^2)_{PP} = 18 - a_P$. Since $M_{Pz} = 0$ for a free P and M is symmetric across equal-size orbits, the diagonal entry expands exactly as

$$(M^2)_{PP} = \sum_k M_{Pk} M_{kP} = a_P^2 + \sum_{Q \neq P,\ Q \text{ free}} f_{PQ}^2 + \sum_{R \in T_1} M_{PR}^2,$$

the three terms collecting the $k = P$, k free $\neq P$, and $k \in T_1$ contributions (the $k \in F$ terms vanish, as $M_{Pz} = 0$). Equating with $18 - a_P$:

$$a_P^2 + \sum_{Q \neq P} f_{PQ}^2 + \sum_{R \in T_1} M_{PR}^2 = 18 - a_P.$$

Since $a_P \in \{0, 2\}$ and $\sum_{R \in T_1} M_{PR} = 6$ with $M_{PR} \in \{0, 1, 2\}$ (Lemma 5.3), one has $\sum_{R \in T_1} M_{PR}^2 = 6 + 2n_2(P)$ where $n_2(P) = \#\{R \in T_1 : M_{PR} = 2\}$. Summing over P and writing $N := \sum_{P \in F_0} n_2(P)$ and $S_2 := \sum_{P<Q \text{ free}} f_{PQ}^2$, this gives

$$2S_2 = \sum_{P \in F_0} \bigl(18 - a_P - a_P^2\bigr) - \sum_{P \in F_0} \sum_{R \in T_1} M_{PR}^2 = (360 - 2t - 4t) - (120 + 2N),$$

that is, $S_2 = 120 - 3t - N$. Hence the identity $S_2 = 108 - 3t$ asserted in earlier drafts is *equivalent* to the single combinatorial statement $N = 12$, where N counts the pairs (P, R) (free orbit, type-1 orbit) with quotient entry $M_{PR} = 2$. We were unable to derive $N = 12$ from the axioms, and we do not use it: no result of this paper depends on the second identity. (Independently, the case is empty, so the question is moot for existence — see §8.)

Remark 5.7. The unconditional content of §5 is $t = 2$ (Corollary 5.4), in a case known to contain no graph [1, 2]. This replaces the "conditional $t = 2$" of version 0.1.0, which rested on the unproved second identity; the sharpening now uses only Lemma 5.3(b) and Theorem 5.2.

6 Involutions

Theorem 6.1 (fixed-point classification; cited)**.** *Let g be an automorphism of prime order p of a putative Γ. Then* $\mathrm{Fix}(g)$ *is a single vertex ($p = 2$ or 7), empty ($p = 3$ or 11), or a triangle ($p = 3$) [6]. In particular an involution ($p = 2$) fixes exactly one vertex.*

Status. This is the classification of fixed-point subgraphs of prime-order automorphisms due to Makhnev and Minakova [6], reproduced as Theorem 1 of Makhnev's survey [5]. It is a statement about the *fixed-point subgraph*, and is distinct from the companion result of the same authors that if $\mathrm{Aut}(\Gamma)$ contains an involution then its order divides 42; the two must not be conflated. The involution instance ($p = 2$) is what we import, and it is used strictly as an input, never as a consequence of integrality alone. The classification is corroborated by secondary reproductions (Behbahani's thesis, Theorem 1.6; Cesarz–Woldar cite Makhnev–Minakova Theorem 2.7 for $|G| \mid 2 \cdot 3^3 \cdot 7 \cdot 11$), but the primary text was not accessible to this revision (only its abstract is public; the exact theorem number and page were not checked). Given this input, the character computation of Theorem 6.2 is self-contained. If the classification is not accepted, Theorem 6.2 should be read as conditional on it.

Theorem 6.2. *Assume Classification 6.1, so that* $\mathrm{Fix}(\tau) = \{z\}$ *and, in the notation of Definition 2.1, $\alpha_0 = 1$, $\alpha_1 = 2e$. Then $e \equiv 0 \pmod 7$ and $e \equiv 1 \pmod 2$, hence $e \in \{7, 21, 35, 49\}$, and the number n of τ-invariant triangles satisfies $n \in \{1, 3, 5, 7\}$ with $n \leq e$.*

Proof. Congruence. $\chi_2(\tau) = \mathrm{tr}(\tau E_3) = \frac{4\alpha_0 + \alpha_1 - 18}{7} = \frac{2(e-7)}{7}$. On the 3-eigenspace τ has eigenvalues ± 1 with multiplicities $p, 54 - p$, so $\chi_2(\tau) = 2p - 54$; equating gives $p = (e + 182)/7 \in \mathbb{Z}$, i.e. $e \equiv 0 \pmod 7$.

Invariant triangles. A τ-invariant triangle has its vertex set permuted by τ. Since z is the only fixed vertex, the set must be $\{z, x, \tau x\}$: it contains z together with one full 2-orbit $\{x, \tau x\}$, and

it is a triangle exactly when $x \sim \tau x$ and z is adjacent to both x and τx. Conversely each edge 2-orbit adjacent to z gives exactly one such triangle. Hence n equals the number of edge 2-orbits adjacent to z. Since the remaining triangles fall into τ-orbits of size 2, $231 = n + 2m'$, so n is odd. Finally z has degree 14 and is adjacent to whole 2-orbits, hence to at most 7 of them, so $n \le 7$; thus $n \in \{1, 3, 5, 7\}$ and $n \le e$. The 693 edges split into the e edges inside 2-orbits and an even number of edges between orbits (no between-orbit edge is fixed by τ), so e is odd; with $e \equiv 0 \pmod 7$ therefore $e \in \{7, 21, 35, 49\}$. □

Remark 6.3. The congruence $e \equiv 0 \pmod 7$ is the integrality condition of [6], made explicit as a congruence on e; it is included for completeness of the trace computation.

7 Computational checks and reproducibility

Status of the accompanying repository. The repository named in earlier drafts (`https://github.com/YesterdaysLemon/conway-99-research`, commit `85e705cc6c2a14d123120c93a847e30aaab1789e`) was located and inspected. It is an `EXPLORATORY` project (its own metadata: "resolution claim: none"; a literature audit through 2026-07-29), comprising roughly 5000 files. Its real entry points are an independent strongly-regular-graph certificate validator `verification/check_srg.py` with fixtures, a `unittest` suite (`python -m unittest discover -s verification -p "test_*.py"`), and a search-only dependency file `requirements-search.txt` (`python-sat`).

Correction to earlier drafts. Earlier drafts of this work described a package `conway99` with files `pyproject.toml`, `scripts/verify_derived_results.py`, `scripts/reproduce_computations.py`, `results/reproducibility-report.md`, `TRACEABILITY.md`, and `AUDIT.md`, and commands `pip install -e ".[dev]"` and `pytest`. **None of these files exists in the repository**, and no package `conway99` is present; those claims are **withdrawn**.

What was run. The repository's calibration validator was executed on its known positive fixture $\mathrm{srg}(9, 4, 1, 2)$,

```
python3 verification/check_srg.py candidates/calibration-srg-9.srg.json \
        --vertices 9 --degree 4 --lambda 1 --mu 2
```

and reported `valid: true` (`combinatorial`, `matrix_identity`). This establishes only that the repository ships a working certificate checker. Independently, the arithmetic checks of Appendix A were run; they reproduce the arithmetic, not the hypotheses.

What remains unverified. The repository does not contain this paper's derivations (the identity $\sum_{P<Q} f_{PQ} = 80 - t$ does not occur in it) and does not reproduce this paper's results. The imported classification (Theorem 6.1) is not checked by any repository. No claim of independent computational verification of this paper's arguments is made, and the reproducibility of the paper's *results* remains unestablished. A passing calibration on Paley(9) is a sanity check of the general congruence, not a proof of it.

8 Related work, limitations and open questions

Related work and novelty. The classical admissible set for the fixed-point-free case is $\{6, 13, 20, 27\}$; the congruence $T \equiv 6 \pmod 7$ follows from the character method of [6] (re-derived above), and the admissible set is a consequence of that congruence together with $T \le 33$. The

refinement to $\{6, 27\}$ via the mod-3 congruence is elementary; whether it is recorded in the orbit-matrix literature [1, 2] is not established here. The order-3 fixed-triangle case is eliminated in [1] (Theorem 4.14 gives zero orbit matrices); Proposition 5.5 and Corollary 5.4 refine its internal parameter ($t = 2$). The involution congruence restates the integrality condition of [6]. The primary texts of [6] and [2] were not consulted in full.

Recent work appears to settle the novelty question negatively. A 2026 dataset [8] is titled around the $T = 27$ branch, but its *abstract* states that its Lean 4 library proves the full two-value result — that any fixed-point-free order-3 automorphism has triangle count $T \in \{6, 27\}$; community orbit-matrix work independently reports the same "6 or 27" split. We could not inspect the dataset's artifacts, so we record this as *reported*, not verified. We therefore do *not* claim $T \in \{6, 27\}$ as new; the contribution of this paper is a clean elementary re-derivation, not a new result.

Limitations and open questions.

1. Existence of Γ is open; nothing here resolves it.
2. The second quotient identity $S_2 = 108 - 3t$ is not derived (it is equivalent to $N = 12$, Remark 5.6); no result depends on it.
3. Classification 6.1 is cited, not verified; Theorem 6.2 is conditional on it.
4. Novelty of $T \in \{6, 27\}$ is unestablished.
5. The values $T = 6$ and $T = 27$, the order-2 case, and the asymmetric case are untouched.
6. The accompanying repository does not contain the files named in earlier drafts (those claims are withdrawn) and does not reproduce this paper's results (§7).

9 Conclusion

The triangle-orbit congruence (Theorem 3.1) together with the spectral projector yields, conditional on the stated hypotheses and imported results, the following necessary conditions for the surviving automorphism cases: $T \in \{6, 27\}$ (fixed-point-free, Theorem 4.1), $t = 2$ (three fixed vertices, Corollary 5.4), and $e \in \{7, 21, 35, 49\}$ (involutions, Theorem 6.2). These reduce the search space but do not settle existence. The first quotient identity (Proposition 5.5) is proved; the second is not used. Because a putative Γ could be asymmetric, even eliminating every prime-order automorphism case would establish only that Γ is asymmetric, not that it fails to exist.

A Verification script

The arithmetic checks referred to in Appendix A are performed by the following self-contained script (Python 3 with `numpy`); it is reproduced here so the checks are reproducible from this paper alone. Save it as `verify_arithmetic.py` and run `python3 verify_arithmetic.py`.

```
# verify_arithmetic.py -- arithmetic checks for the Conway 99-graph papers.
# Requires Python 3 and numpy. Run:  python3 verify_arithmetic.py
import numpy as np

print("TEST 1 (general): spectrum multiplicities for (99,14,1,2)")
```

```
for f3 in range(99):
    f4 = 98 - f3
    if 14 + 3*f3 - 4*f4 == 0:
        print("  f3 =", f3, "  f-4 =", f4)

print("TEST 2 (general): congruence -> candidate sets")
print("  T<=33, T=6 mod 21 :", [x for x in range(34) if x % 21 == 6])
print("  t<=20, t=2 mod 21 :", [x for x in range(21) if x % 21 == 2])
print("  e<=49, 7|e, odd   :", [x for x in range(50)
                                 if x % 7 == 0 and x % 2 == 1])

print("TEST 3 (general): triangle count")
edges = 99*14//2
print("  edges =", edges, " triangles =", edges//3)

print("TEST 4 (known example): quotient identity on Paley(9)")
def add(x,y): return ((x[0]+y[0])%3,(x[1]+y[1])%3)
def mul(x,y): return ((x[0]*y[0]-x[1]*y[1])%3,(x[0]*y[1]+x[1]*y[0])%3)
def neg(x):   return ((-x[0])%3,(-x[1])%3)
def sub(x,y): return add(x,neg(y))
def pw(x,k):
    r=(1,0)
    for _ in range(k): r=mul(r,x)
    return r
els=[(a,b) for a in range(3) for b in range(3)]
idx={e:i for i,e in enumerate(els)}
sq=sorted({pw(x,2) for x in els if x!=(0,0)})
A=np.zeros((9,9),dtype=int)
for x in els:
    for y in els:
        if x!=y and sub(x,y) in sq: A[idx[x],idx[y]]=1
cands=[]
for a in sq:
    for b in els:
        for p in (1,3):
            perm=[idx[add(mul(a,pw(x,p)),b)] for x in els]
            if len(set(perm))==9 and all(
                    A[perm[i],perm[j]]==A[i,j]
                    for i in range(9) for j in range(9)):
                cands.append(tuple(perm))
cands=sorted(set(cands))
def orbs(p):
    seen=set(); o=[]
    for v in range(9):
        if v in seen: continue
        c=[]; u=v
        while u not in c: c.append(u); seen.add(u); u=p[u]
        o.append(sorted(c))
```

```
    return o
allok=True
for p in cands:
    O=orbs(p); OI={}
    for i,o in enumerate(O):
        for v in o: OI[v]=i
    sz=[len(o) for o in O]
    M=np.zeros((len(O),len(O)),dtype=int)
    for i,o in enumerate(O):
        for y in range(9):
            if A[o[0],y]: M[i,OI[y]]+=1
    for i in range(len(O)):
        for j in range(len(O)):
            if (M@M)[i,j] != (2 if i==j else 0) - M[i,j] + 2*sz[j]:
                allok=False
print("  automorphisms:", len(cands), "  identity holds for all:", allok)

print("NOTE: no test here verifies the main result sum f_PQ = 80 - t,")
print("which is a proof about a hypothetical graph, not a computation.")
```

Expected output (observed with Python 3.12 and numpy 2.5):

```
TEST 1 (general): spectrum multiplicities for (99,14,1,2)
  f3 = 54   f-4 = 44
TEST 2 (general): congruence -> candidate sets
  T<=33, T=6 mod 21 : [6, 27]
  t<=20, t=2 mod 21 : [2]
  e<=49, 7|e, odd   : [7, 21, 35, 49]
TEST 3 (general): triangle count
  edges = 693  triangles = 231
TEST 4 (known example): quotient identity on Paley(9)
  automorphisms: 72   identity holds for all: True
NOTE: no test here verifies the main result sum f_PQ = 80 - t,
which is a proof about a hypothetical graph, not a computation.
```

Scope. Tests 1–3 check general arithmetic — the multiplicity equations, the congruence-to-candidate-set step, and the triangle count. Test 4 checks the quotient identity (Lemma 2.7) on the known graph Paley(9). **None of these tests verifies this paper's main result** $\sum_{P<Q} f_{PQ} = 80 - t$: that is a proof about a hypothetical graph, not a computation, and cannot be checked this way. The imported classification (Theorem 6.1) is likewise not checked here.

Author contributions and AI-use disclosure

The mathematical content of this manuscript, its arguments, and its verification code were produced by AI systems, not by the human author. The human author (Prathik Katukala) initiated and commissioned the project and supplied the research prompts; the human author has not independently verified the derivations, references, or computations and does not take responsibility for the mathematical claims.

Verification status. As of this version no human has independently checked any claim below. In particular the following require human verification before any reliance: the adjacency-matrix identity and the projector identities (Lemma 2.2, 2.4); the spectrum (Lemma 2.3); the triangle count (Lemma 2.6); the quotient identity (Lemma 2.7); the three congruences (Theorems 4.1, 5.2, 6.2); the first quotient identity (Proposition 5.5); the cited fixed-point classification (Theorem 6.1); the arithmetic in Appendix A; and every bibliographic entry. Because of this, the manuscript is not, as it stands, suitable for submission: most journals require a human author who takes responsibility for the work, and policies on generative AI differ. The target venue's authorship and AI-disclosure requirements must be read and followed before any submission.

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