Direct-Product Intercalate Count Identities for Latin Squares

Augusto Rehfeldt

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Abstract

This paper records a single counting identity for intercalates in the direct product of Latin squares, obtained in three verified formulations. For Latin squares A of order m and B of order n, with $I(\cdot)$ the number of intercalates and $A \times B$ the direct product of order mn, the identity is $I(A \times B) = n^2 I(A) + m^2 I(B) + 4I(A)I(B)$. The three formulations differ in notation and in the organization of the counting argument but state the same formula. Each formulation was checked by exhaustive enumeration on small orders and by an independent re-derivation of its lemmas. Lean 4 formalizations compile with `sorry` and are not sorry-free. Automated searches against arXiv, Crossref and OpenAlex returned no matching statement. This is a bounded index search, not a literature review.

1. Introduction

1.1 Context

A Latin square of order n is an $n \times n$ array in which each symbol occurs exactly once in each row and each column. An intercalate is a 2×2 Latin subsquare: a pair of distinct rows and a pair of distinct columns whose four entries use exactly two distinct symbols with opposite corners equal. Intercalate enumeration is studied in work on random Latin squares, maximum intercalate numbers, and bounds on small subsquares. Separately, product constructions for Latin squares and quasigroups are studied for orthogonality, subsquare existence, autotopisms, and full and overlapping products.

1.2 What is new

The results proved here give a closed polynomial for the number of intercalates in a direct product in terms of the numbers in the factors. A 2×2 pattern in the product is an intercalate exactly when both coordinate projections have matching diagonals. In a Latin square, matching diagonals on distinct rows and columns force an intercalate, while mixed equal-row with distinct-column and distinct-row with equal-column patterns are impossible. Valid patterns therefore split into three types: intercalate in both factors, intercalate in A with a degenerate constant cell in B, and degenerate constant cell in A with intercalate in B. Counting ordered lifts gives $n^2 I(A)$ and $m^2 I(B)$ for the single-factor types and $4I(A)I(B)$ for the double-intercalate type, the factor 4 coming from two row alignments times two column alignments. The three theorems below are the same identity proved by variants of this case analysis.

2. Notation

A Latin square of order k is used in two equivalent notations across the three formulations: as a $k \times k$ array with entries from $\{0, \dots, k-1\}$ or $[k] = \{1, \dots, k\}$ in which each symbol occurs exactly once in each row and column, and equivalently as a map $L : [k] \times [k] \to [k]$ whose every row and every column is a permutation of $[k]$. Row and column injectivity over finite sets of equal cardinality is equivalent to the exactly-once property.

An intercalate is an unordered pair of distinct rows $\{r_1, r_2\}$ together with an unordered pair of distinct columns $\{c_1, c_2\}$ such that $L(r_1, c_1) = L(r_2, c_2)$ and $L(r_1, c_2) = L(r_2, c_1)$, with $L(r_1, c_1) \neq L(r_1, c_2)$. In a

Latin square the inequality is automatic given distinct rows and columns and the two equalities. $I(L)$, also written $i(L)$, denotes the number of such row-pair column-pair combinations.

The direct product $C = A \times B$ of A of order m and B of order n has row set, column set and symbol set indexed by pairs, of cardinality mn, with entry $(A[a, c], B[b, d])$ at row (a, b) and column (c, d). Up to fixed bijections identifying $[m] \times [n]$ with $[mn]$ this is a Latin square of order mn, and renaming by bijections does not change the intercalate count.

3. Results

Theorem (c1: Direct-product intercalate count identity for Latin squares)**.** *For every pair of integers m >= 1 and n >= 1, and for every Latin square A of order m and every Latin square B of order n, the number of intercalates in the direct product satisfies I(A x B) = n^2 I(A) + m^2 I(B) + 4 I(A) I(B).*

Theorem. *Let $m \geq 1$ and $n \geq 1$ be integers. Let A be a Latin square of order m and let B be a Latin square of order n. Let $A \times B$ denote the $mn \times mn$ array whose rows are indexed by pairs (a, b), whose columns are indexed by pairs (c, d), and whose symbols are pairs (e, f), with $a, c, e \in \{0, \dots, m-1\}$ and $b, d, f \in \{0, \dots, n-1\}$, defined by*

$$(A \times B)[(a, b), (c, d)] = (A[a, c], B[b, d]).$$

Then

$$I(A \times B) = n^2 I(A) + m^2 I(B) + 4 I(A) I(B),$$

where $I(L)$ denotes the number of intercalates in L.

For this proof, an intercalate is understood as follows. Given a total order on row indices and on column indices, an intercalate is a choice of distinct rows $r_1 < r_2$ and distinct columns $s_1 < s_2$ such that, writing $w_{ij} = L[r_i, s_j]$, one has $w_{11} = w_{22}$, $w_{12} = w_{21}$, and $w_{11} \neq w_{12}$. Hence the four positions contain exactly two distinct symbols, with each diagonal constant. Since each unordered pair of distinct rows corresponds to exactly one ordered pair satisfying $<$, and likewise for columns, $I(L)$ can equivalently be counted as the number of unordered pairs $\{r_1, r_2\}$ and $\{s_1, s_2\}$ whose 2×2 submatrix has this form.

Lemma 1. *The array $A \times B$ is a Latin square of order mn.*

Proof of Lemma 1. The set of rows has cardinality mn, the set of columns has cardinality mn, and the set of symbols $\{0, \dots, m-1\} \times \{0, \dots, n-1\}$ has cardinality mn. Fix a row (a, b) and two distinct columns $(c_1, d_1) \neq (c_2, d_2)$. Suppose for contradiction that

$$(A[a, c_1], B[b, d_1]) = (A[a, c_2], B[b, d_2]).$$

Then $A[a, c_1] = A[a, c_2]$ and $B[b, d_1] = B[b, d_2]$. Since the columns are distinct, either $c_1 \neq c_2$ or $d_1 \neq d_2$. If $c_1 \neq c_2$, then row a of A contains the same symbol in two distinct columns, contradicting the hypothesis that A is Latin. If $d_1 \neq d_2$, then row b of B contains the same symbol in two distinct columns, contradicting the hypothesis that B is Latin. Therefore distinct columns receive distinct symbols in each fixed row.

Fix a column (c, d) and two distinct rows $(a_1, b_1) \neq (a_2, b_2)$. Suppose

$$(A[a_1, c], B[b_1, d]) = (A[a_2, c], B[b_2, d]).$$

Then $A[a_1, c] = A[a_2, c]$ and $B[b_1, d] = B[b_2, d]$. Since the rows are distinct, either $a_1 \neq a_2$ or $b_1 \neq b_2$. If $a_1 \neq a_2$, column c of A repeats a symbol, which is impossible. If $b_1 \neq b_2$, column d of B repeats a symbol, which is impossible. Therefore distinct rows receive distinct symbols in each fixed column. Since each row and each column contains mn pairwise distinct symbols drawn from a set of mn symbols, each symbol occurs exactly once in each row and each column. Hence $A \times B$ is Latin of order mn.

□

Lemma 2. *Let L be a Latin square. Let $r_1 \neq r_2$ be distinct rows and $s_1 \neq s_2$ be distinct columns. Then the unordered pairs $\{r_1, r_2\}, \{s_1, s_2\}$ form an intercalate if and only if*

$$L[r_1, s_1] = L[r_2, s_2] \quad \textit{and} \quad L[r_1, s_2] = L[r_2, s_1].$$

Proof of Lemma 2. Assume both equalities hold. Because $s_1 \neq s_2$, row r_1 of L has distinct entries in columns s_1 and s_2, so $L[r_1, s_1] \neq L[r_1, s_2]$. By the assumed equalities, the two diagonal values are distinct. Opposite corners are equal by assumption, and exactly two distinct symbols occur. Hence the pairs form an intercalate.

Conversely, assume the pairs form an intercalate, meaning that with $w_{ij} = L[r_i, s_j]$ one has $w_{11} = w_{22}$, $w_{12} = w_{21}$, and $w_{11} \neq w_{12}$. Then in particular both displayed equalities hold. Moreover, if only the definition requiring exactly two symbols and equality on at least one diagonal were used, equality on both diagonals would still follow: for example, if $L[r_1, s_1] = L[r_2, s_2] = x$ and only two symbols occur, then $L[r_2, s_1] \neq x$ by distinctness of column s_1 entries in distinct rows, so $L[r_2, s_1] = y$ for the second symbol $y \neq x$; similarly $L[r_1, s_2] \neq x$ by distinctness of row r_1 entries, so $L[r_1, s_2] = y$, yielding equality on the second diagonal as well. The same argument applies starting from the other diagonal.

□

Lemma 3. *Let L be a Latin square of order t. Let r_1, r_2 be rows, possibly equal, and s_1, s_2 be columns, possibly equal. Consider the condition*

$$L[r_1, s_1] = L[r_2, s_2] \quad \textit{and} \quad L[r_1, s_2] = L[r_2, s_1].$$

If $r_1 = r_2$ and $s_1 = s_2$, the condition always holds. If $r_1 = r_2$ and $s_1 \neq s_2$, the condition never holds. If $r_1 \neq r_2$ and $s_1 = s_2$, the condition never holds.

Proof of Lemma 3. If $r_1 = r_2 = r$ and $s_1 = s_2 = s$, both equalities reduce to $L[r, s] = L[r, s]$, which holds.

If $r_1 = r_2 = r$ and $s_1 \neq s_2$, the first equality asserts $L[r, s_1] = L[r, s_2]$, which contradicts the property that row r has distinct symbols in distinct columns. Hence both equalities cannot hold jointly.

If $r_1 \neq r_2$ and $s_1 = s_2 = s$, the first equality asserts $L[r_1, s] = L[r_2, s]$, which contradicts the property that column s has distinct symbols in distinct rows. Hence both equalities cannot hold jointly.

□

Lemma 4. *Let A be of order m. Let S_A be the set of quadruples (a_1, a_2, c_1, c_2) with $a_1, a_2, c_1, c_2 \in \{0, \ldots, m-1\}$ satisfying*

$$A[a_1, c_1] = A[a_2, c_2] \quad \textit{and} \quad A[a_1, c_2] = A[a_2, c_1].$$

Then $|S_A| = m^2 + 4I(A)$. Similarly, for B of order n, the set S_B of quadruples (b_1, b_2, d_1, d_2) satisfying the analogous two equalities has cardinality $n^2 + 4I(B)$.

Proof of Lemma 4. Partition S_A according to whether $a_1 = a_2$ and whether $c_1 = c_2$. By Lemma 3, if $a_1 = a_2$ but $c_1 \neq c_2$, no quadruple satisfies the equalities, and if $a_1 \neq a_2$ but $c_1 = c_2$, no quadruple satisfies the equalities. If $a_1 = a_2$ and $c_1 = c_2$, every quadruple satisfies the equalities, and there are $m \cdot m = m^2$ such quadruples, obtained by choosing a_1 and c_1 arbitrarily and setting $a_2 = a_1$, $c_2 = c_1$.

It remains to count quadruples with $a_1 \neq a_2$ and $c_1 \neq c_2$ satisfying the equalities. By Lemma 2, for distinct rows and distinct columns, the equalities hold exactly when the unordered pairs $\{a_1, a_2\}, \{c_1, c_2\}$ form an intercalate in A. Each intercalate, counted in $I(A)$ as one unordered row pair and one unordered column pair, corresponds to exactly four ordered quadruples (a_1, a_2, c_1, c_2) with $a_1 \neq a_2$ and $c_1 \neq c_2$: there are two orders for the two distinct rows and two orders for the two distinct columns, and the equalities are preserved under swapping a_1 with a_2 and under swapping c_1 with c_2. Conversely every ordered quadruple with distinct rows and distinct columns satisfying the equalities arises in this way from a unique unordered intercalate. Therefore the number of such ordered quadruples is $4I(A)$.

Adding the two non-empty cases gives $|S_A| = m^2 + 4I(A)$. The same reasoning applied to B gives $|S_B| = n^2 + 4I(B)$.

□

Proof of the Theorem. Fix arbitrary total orders on the row index set and column index set of $A \times B$ so that $I(A \times B)$ is defined. By Lemma 1, $A \times B$ is Latin, so Lemma 2 applies to it. Hence each unordered intercalate corresponds to exactly four ordered quadruples (R_1, R_2, C_1, C_2) with $R_1 \neq R_2$ and $C_1 \neq C_2$ satisfying

$$(A \times B)[R_1, C_1] = (A \times B)[R_2, C_2] \quad \text{and} \quad (A \times B)[R_1, C_2] = (A \times B)[R_2, C_1],$$

and every such ordered quadruple arises from a unique unordered intercalate. Consequently the number of ordered quadruples with distinct product rows and distinct product columns satisfying these two equalities equals $4I(A \times B)$.

Write $R_i = (a_i, b_i)$ and $C_j = (c_j, d_j)$. Since a pair of symbols is equal if and only if both coordinates are equal, the first product equality holds if and only if $A[a_1, c_1] = A[a_2, c_2]$ and $B[b_1, d_1] = B[b_2, d_2]$, and the second holds if and only if $A[a_1, c_2] = A[a_2, c_1]$ and $B[b_1, d_2] = B[b_2, d_1]$. Regrouping, the two product equalities hold jointly if and only if $(a_1, a_2, c_1, c_2) \in S_A$ and $(b_1, b_2, d_1, d_2) \in S_B$, with S_A, S_B as in Lemma 4.

Therefore ordered product quadruples satisfying the diagonal equalities are in bijection with pairs in $S_A \times S_B$, except for the requirement that product rows be distinct and product columns be distinct. The Cartesian product has cardinality

$$|S_A \times S_B| = (m^2 + 4I(A))(n^2 + 4I(B)).$$

Among these, some have $R_1 = R_2$ or $C_1 = C_2$ and must be removed.

Suppose $(a_1, a_2, c_1, c_2) \in S_A$ and $(b_1, b_2, d_1, d_2) \in S_B$ with $R_1 = R_2$, meaning $a_1 = a_2$ and $b_1 = b_2$. Since $(a_1, a_2, c_1, c_2) \in S_A$ with $a_1 = a_2$, Lemma 3 forces $c_1 = c_2$. Since $(b_1, b_2, d_1, d_2) \in S_B$ with $b_1 = b_2$, Lemma 3 forces $d_1 = d_2$. Hence $C_1 = C_2$ as well, and the full quadruple is constant in each coordinate: $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$. Similarly, if $C_1 = C_2$, meaning $c_1 = c_2$ and $d_1 = d_2$, Lemma 3 applied to S_A forces $a_1 = a_2$ and applied to S_B forces $b_1 = b_2$, so $R_1 = R_2$. Thus within $S_A \times S_B$, the condition $R_1 = R_2$ and the condition $C_1 = C_2$ are equivalent, and both are equivalent to full constancy in each coordinate pair.

The number of fully constant tuples is determined by independent choices of $a_1 \in \{0, \ldots, m-1\}$, $b_1 \in \{0, \ldots, n-1\}$, $c_1 \in \{0, \ldots, m-1\}$, $d_1 \in \{0, \ldots, n-1\}$, giving $mnmn = m^2n^2$ tuples. Each such tuple belongs to $S_A \times S_B$ by Lemma 3. Hence exactly m^2n^2 elements of $S_A \times S_B$ fail the distinctness requirement $R_1 \neq R_2$ and $C_1 \neq C_2$.

Subtracting yields the number of ordered quadruples with distinct product rows and distinct product columns satisfying the product diagonal equalities:

$$4I(A \times B) = (m^2 + 4I(A))(n^2 + 4I(B)) - m^2n^2.$$

Expanding the product gives $m^2n^2 + 4n^2I(A) + 4m^2I(B) + 16I(A)I(B)$, and subtracting m^2n^2 leaves $4n^2I(A) + 4m^2I(B) + 16I(A)I(B)$. Dividing by 4 gives

$$I(A \times B) = n^2I(A) + m^2I(B) + 4I(A)I(B),$$

as required. This argument uses no assumption on m or n beyond $m \geq 1$ and $n \geq 1$; when $m = 1$ or $n = 1$ there are no pairs of distinct rows or columns in the corresponding factor, so $I = 0$ there, and the counting above remains valid with the corresponding $4I$ term equal to zero.

□

Theorem (c2: Intercalate count of the direct product of two Latin squares)**.** *For every pair of integers m >= 1 and n >= 1, for every Latin square A of order m and every Latin square B of order n, letting C be the direct product A x B of order m*n, the number I(C) of intercalates in C satisfies I(C) = 4*I(A)*I(B) + (n^2)*I(A) + (m^2)*I(B).*

Theorem. *Let $m \geq 1$ and $n \geq 1$ be integers. Let $A : [m] \times [m] \to [m]$ be a Latin square of order m and $B : [n] \times [n] \to [n]$ be a Latin square of order n. Let C be the direct product $A \times B$, with row set $[m] \times [n]$, column set $[m] \times [n]$, and symbol set $[m] \times [n]$, defined by*

$$C((r_A, r_B), (c_A, c_B)) = (A(r_A, c_A), B(r_B, c_B)).$$

Then, with $I(\cdot)$ denoting the number of intercalates,

$$I(C) = 4\, I(A)\, I(B) + n^2\, I(A) + m^2\, I(B).$$

Recall that $[k] = \{1, \ldots, k\}$, that a Latin square of order k is a map $L : [k] \times [k] \to [k]$ whose every row and every column is a permutation of $[k]$, and that an intercalate is an unordered pair of distinct rows $\{r_1, r_2\}$ together with an unordered pair of distinct columns $\{c_1, c_2\}$ such that, writing the rows as r_1, r_2 and the columns as c_1, c_2 in some order, $L(r_1, c_1) = L(r_2, c_2)$ and $L(r_1, c_2) = L(r_2, c_1)$ with $L(r_1, c_1) \neq L(r_1, c_2)$.

Lemma 1 (The product is Latin and counting is labelling-invariant)**.** *Up to renaming of rows, columns, and symbols, C is a Latin square of order mn, and its number of intercalates does not depend on the chosen renamings.*

Proof of Lemma 1. The set $[m] \times [n]$ has cardinality mn. Fix any bijections from $[m] \times [n]$ to $[mn]$ for rows, columns, and symbols. Transporting C along these bijections yields a map $[mn] \times [mn] \to [mn]$. Renaming rows, columns, or symbols by bijections sends intercalates to intercalates bijectively, so the count $I(C)$ is the same for any choice of bijections.

It remains to verify the Latin property. Fix a row $R = (r_A, r_B)$. The map $(c_A, c_B) \mapsto (A(r_A, c_A), B(r_B, c_B))$ is the Cartesian product of the map $c_A \mapsto A(r_A, c_A)$ and the map $c_B \mapsto B(r_B, c_B)$. Since A is Latin, $c_A \mapsto A(r_A, c_A)$ is a permutation of $[m]$. Since B is Latin, $c_B \mapsto B(r_B, c_B)$ is a permutation of $[n]$. The product of two permutations is a permutation of the product set. Hence every row of C contains each symbol of $[m] \times [n]$ exactly once. The same argument applied with rows and columns exchanged shows that for fixed (c_A, c_B), the map $(r_A, r_B) \mapsto (A(r_A, c_A), B(r_B, c_B))$ is a permutation of $[m] \times [n]$. Hence every column is a permutation. Therefore C is Latin.

□

Henceforth rows, columns, and symbols of C are written as pairs, and intercalates in C are pairs $\{R_1, R_2\}$, $\{K_1, K_2\}$ with $R_1 \neq R_2$ in $[m] \times [n]$ and $K_1 \neq K_2$ in $[m] \times [n]$.

Lemma 2 (Diagonal-matching in a Latin square)**.** *Let $L : [k] \times [k] \to [k]$ be Latin. Let $x_1, x_2 \in [k]$ be rows, possibly equal, and $y_1, y_2 \in [k]$ be columns, possibly equal. Suppose the diagonal-matching equalities hold:*

$$L(x_1, y_1) = L(x_2, y_2) \quad \text{and} \quad L(x_1, y_2) = L(x_2, y_1).$$

Then either $x_1 = x_2$ and $y_1 = y_2$, in which case the equalities hold automatically, or $x_1 \neq x_2$ and $y_1 \neq y_2$. The mixed cases with exactly one of $x_1 = x_2$, $y_1 = y_2$ holding are impossible. Moreover, when $x_1 \neq x_2$ and $y_1 \neq y_2$, diagonal-matching implies $L(x_1, y_1) \neq L(x_1, y_2)$, so $\{x_1, x_2\}, \{y_1, y_2\}$ is an intercalate.

Proof of Lemma 2. If $x_1 = x_2$ and $y_1 = y_2$, both equalities reduce to identities.

Suppose $x_1 = x_2 =: x$ while $y_1 \neq y_2$. Then the first equality gives $L(x, y_1) = L(x, y_2)$. This states that row x contains the same symbol in two distinct columns, contradicting the hypothesis that every row of L is a permutation. Hence this combination cannot occur under the diagonal-matching hypothesis.

Suppose $x_1 \neq x_2$ while $y_1 = y_2 =: y$. Then the first equality gives $L(x_1, y) = L(x_2, y)$. This states that column y contains the same symbol in two distinct rows, contradicting the hypothesis that every column of L is a permutation. Hence this combination cannot occur either.

Therefore diagonal-matching implies that equality of the two rows is equivalent to equality of the two columns: either both pairs are equal or both pairs are distinct.

Suppose now $x_1 \neq x_2$ and $y_1 \neq y_2$ with diagonal-matching. If in addition $L(x_1, y_1) = L(x_1, y_2)$, then row x_1 repeats a symbol in distinct columns y_1, y_2, contradicting the Latin property. Hence $L(x_1, y_1) \neq L(x_1, y_2)$. Combined with the two diagonal equalities, this is exactly the definition of an intercalate with rows $\{x_1, x_2\}$ and columns $\{y_1, y_2\}$. Note that interchanging the names $x_1 \leftrightarrow x_2$ interchanges the two equalities, and interchanging $y_1 \leftrightarrow y_2$ also interchanges the two equalities, so the property depends only on the unordered pairs.

□

Lemma 3 (Classification of intercalates in the product)**.** *Let $R_1 = (r_{A1}, r_{B1})$, $R_2 = (r_{A2}, r_{B2})$ be distinct rows of C and $K_1 = (c_{A1}, c_{B1})$, $K_2 = (c_{A2}, c_{B2})$ be distinct columns of C. Then $\{R_1, R_2\}, \{K_1, K_2\}$ is an intercalate in C if and only if the following three conditions hold:*

(i) the A-projection is diagonal-matching, namely $A(r_{A1}, c_{A1}) = A(r_{A2}, c_{A2})$ and $A(r_{A1}, c_{A2}) = A(r_{A2}, c_{A1})$;

(ii) the B-projection is diagonal-matching, namely $B(r_{B1}, c_{B1}) = B(r_{B2}, c_{B2})$ and $B(r_{B1}, c_{B2}) = B(r_{B2}, c_{B1})$;

(iii) the product inequality holds, namely $C(R_1, K_1) \neq C(R_1, K_2)$, i.e. $A(r_{A1}, c_{A1}) \neq A(r_{A1}, c_{A2})$ or $B(r_{B1}, c_{B1}) \neq B(r_{B1}, c_{B2})$ (or both).

Moreover, every intercalate in C belongs to exactly one of the following three types, where a denotes the statement $r_{A1} = r_{A2}$ and b denotes the statement $r_{B1} = r_{B2}$:

- *Type BB: a is false and b is false; the A-projection is an intercalate in A and the B-projection is an intercalate in B.*
- *Type CB: a is true and b is false; thus $r_{A1} = r_{A2}$, $c_{A1} = c_{A2}$, the A-projection is constant, and the B-projection is an intercalate in B.*
- *Type BC: a is false and b is true; thus $r_{B1} = r_{B2}$, $c_{B1} = c_{B2}$, the B-projection is constant, and the A-projection is an intercalate in A.*

No other pattern of equalities is possible for an intercalate in C.

Proof of Lemma 3. Equality in $[m] \times [n]$ holds if and only if it holds in both coordinates. Applied to $C(R_1, K_1) = C(R_2, K_2)$, this gives one A-equality and one B-equality. Applied to $C(R_1, K_2) = C(R_2, K_1)$, this gives the other A-equality and the other B-equality. The inequality $C(R_1, K_1) \neq C(R_1, K_2)$ means the ordered pairs differ in at least one coordinate, which is the stated disjunction. Hence intercalates in C are characterized by (i)–(iii).

Assume (i) and (ii). By Lemma 2 applied to A, (i) forces $r_{A1} = r_{A2}$ if and only if $c_{A1} = c_{A2}$, and if both are distinct then the A-projection is an intercalate. By Lemma 2 applied to B, (ii) forces $r_{B1} = r_{B2}$ if and only if $c_{B1} = c_{B2}$, and if both are distinct then the B-projection is an intercalate.

Since $R_1 \neq R_2$, it is not the case that $r_{A1} = r_{A2}$ and $r_{B1} = r_{B2}$ both hold. In the notation above, $\neg(a \wedge b)$. Since $K_1 \neq K_2$, it is not the case that $c_{A1} = c_{A2}$ and $c_{B1} = c_{B2}$ both hold, which by the equivalences just established is the same condition $\neg(a \wedge b)$. Hence $(a, b) \neq (\text{true}, \text{true})$. The three remaining truth assignments give exactly the three listed types. The mixed degenerate possibilities within a single coordinate, where row-equality and column-equality differ, have been excluded by Lemma 2.

□

Lemma 4 (Constant-A type)**.** *The number of intercalates in C of Type CB equals $m^2\, I(B)$.*

Proof of Lemma 4. Fix $r_A^\star \in [m]$ and $c_A^\star \in [m]$. Consider row pairs of the form $R_1 = (r_A^\star, r_{B1})$, $R_2 = (r_A^\star, r_{B2})$ with $r_{B1} \neq r_{B2}$, and column pairs of the form $K_1 = (c_A^\star, c_{B1})$, $K_2 = (c_A^\star, c_{B2})$ with $c_{B1} \neq c_{B2}$. For such pairs, the A-projection has $r_{A1} = r_{A2} = r_A^\star$ and $c_{A1} = c_{A2} = c_A^\star$, so condition (i) of Lemma 3 holds automatically with common value $A(r_A^\star, c_A^\star)$. Condition (ii) becomes exactly the diagonal-matching condition for $\{r_{B1}, r_{B2}\}, \{c_{B1}, c_{B2}\}$ in B. By Lemma 2, for distinct rows and columns this is equivalent to

being an intercalate in B. Condition (iii) requires $A(r_A^\star, c_A^\star) \neq A(r_A^\star, c_A^\star)$ or $B(r_{B1}, c_{B1}) \neq B(r_{B1}, c_{B2})$; the first disjunct is false, while the second holds whenever the B-projection is an intercalate, because distinct columns in a row of the Latin square B carry distinct symbols. Hence $\{R_1, R_2\}, \{K_1, K_2\}$ is an intercalate in C if and only if $\{r_{B1}, r_{B2}\}, \{c_{B1}, c_{B2}\}$ is an intercalate in B.

Thus for each fixed $(r_A^\star, c_A^\star)$ there is a bijection between intercalates of this restricted form and intercalates in B. There are m choices for $r_A^\star$ and m choices for $c_A^\star$, giving $m^2\, I(B)$ intercalates counted with those fixed coordinates.

Every Type CB intercalate arises in this way for a unique $(r_A^\star, c_A^\star)$, namely the common A-row and common A-column identified in Lemma 3. Different choices of $(r_A^\star, c_A^\star)$ give different row sets or column sets in C, hence disjoint families. Therefore the total number of Type CB intercalates is $m^2\, I(B)$.

□

Lemma 5 (Constant-B type)**.** *The number of intercalates in C of Type BC equals $n^2\, I(A)$.*

Proof of Lemma 5. The argument is symmetric to Lemma 4 with the roles of A and B exchanged. Fix $r_B^\star \in [n]$ and $c_B^\star \in [n]$. Rows $R_1 = (r_{A1}, r_B^\star)$, $R_2 = (r_{A2}, r_B^\star)$ with $r_{A1} \neq r_{A2}$ and columns $K_1 = (c_{A1}, c_B^\star)$, $K_2 = (c_{A2}, c_B^\star)$ with $c_{A1} \neq c_{A2}$ have constant B-projection, so condition (ii) of Lemma 3 holds automatically. Condition (i) is then equivalent to $\{r_{A1}, r_{A2}\}, \{c_{A1}, c_{A2}\}$ being an intercalate in A by Lemma 2. Condition (iii) follows from the A-inequality $A(r_{A1}, c_{A1}) \neq A(r_{A1}, c_{A2})$, which holds for any intercalate in A, since differing first coordinates already make the ordered pairs $C(R_1, K_1)$ and $C(R_1, K_2)$ distinct. Hence for each fixed $(r_B^\star, c_B^\star)$ there are exactly $I(A)$ intercalates of this form. There are n^2 choices of $(r_B^\star, c_B^\star)$, and every Type BC intercalate arises uniquely in this way. The total is $n^2\, I(A)$.

□

Lemma 6 (Double-intercalate type)**.** *The number of intercalates in C of Type BB equals $4\, I(A)\, I(B)$.*

Proof of Lemma 6. Define the projection map π from Type BB intercalates in C to pairs of intercalates (J_A, J_B), where J_A is an intercalate in A and J_B is an intercalate in B, by projecting rows and columns onto each coordinate. If $\{R_1, R_2\}, \{K_1, K_2\}$ has Type BB, then $r_{A1} \neq r_{A2}$, $c_{A1} \neq c_{A2}$, $r_{B1} \neq r_{B2}$, $c_{B1} \neq c_{B2}$, and Lemma 2 applied to the diagonal-matching conditions (i)–(ii) shows that the projections are indeed intercalates. Condition (iii) is then automatic, because the A-projection already satisfies $A(r_{A1}, c_{A1}) \neq A(r_{A1}, c_{A2})$, so the product pairs differ in the first coordinate. Thus π is well defined.

Fix $J_A = (\{a_1, a_2\}, \{p_1, p_2\})$ and $J_B = (\{b_1, b_2\}, \{q_1, q_2\})$. The preimage under π consists of those $\{R_1, R_2\}, \{K_1, K_2\}$ whose A-projection as unordered pairs equals J_A and whose B-projection equals J_B, with all four coordinate pairs distinct.

Consider the set $\{a_1, a_2\} \times \{b_1, b_2\}$, which has four elements. A two-element subset $\{R_1, R_2\}$ projects to $\{a_1, a_2\}$ in the first coordinate and to $\{b_1, b_2\}$ in the second coordinate if and only if its two elements differ in both coordinates, i.e. the subset is one of the two diagonals of the 2×2 grid. Explicitly, up to ordering of R_1, R_2, the possibilities are $\{(a_1, b_1), (a_2, b_2)\}$ and $\{(a_1, b_2), (a_2, b_1)\}$. No other two-element subset has both projections of size two: any subset contained in a single row $\{a\} \times \{b_1, b_2\}$ has singleton A-projection, any subset contained in a single column has singleton B-projection, and a singleton repeated is excluded by distinctness. Hence there are exactly two possible unordered row pairs lifting (J_A, J_B).

The same reasoning applied to $\{p_1, p_2\} \times \{q_1, q_2\}$ shows that there are exactly two possible unordered column pairs, namely $\{(p_1, q_1), (p_2, q_2)\}$ and $\{(p_1, q_2), (p_2, q_1)\}$.

Combining the two independent choices gives four candidate pairs $(\{R_1, R_2\}, \{K_1, K_2\})$. Each candidate satisfies (i) and (ii): indeed, writing representatives e.g. $R_1 = (a_1, b_1)$, $R_2 = (a_2, b_2)$, $K_1 = (p_1, q_1)$, $K_2 = (p_2, q_2)$, the A-diagonal equalities hold because J_A is an intercalate, and the B-diagonal equalities hold because J_B is an intercalate; the other three combinations use the same values with one or both of the B-coordinates cross-matched, and the B-intercalate identities $B(b_1, q_1) = B(b_2, q_2)$ and $B(b_1, q_2) = B(b_2, q_1)$ guarantee matching in every combination, while the A-identities are unaffected by the B-alignment. Since

unordered interchange of K_1, K_2 or R_1, R_2 preserves the two equalities, the verification does not depend on the chosen representatives. Condition (iii) holds because the A-symbols on the two columns differ. Hence all four candidates are intercalates in C, all have Type BB, and all project to (J_A, J_B).

Moreover, distinct candidates have distinct row sets or distinct column sets, so they are distinct intercalates. No other Type BB intercalate projects to (J_A, J_B), by the enumeration of lifts. Therefore each fibre of π has exactly four elements.

Since there are $I(A)\,I(B)$ pairs (J_A, J_B), the total number of Type BB intercalates is $4\,I(A)\,I(B)$.

□

Proof of the theorem. By Lemma 1, C is Latin up to the fixed identification of $[m] \times [n]$ with $[mn]$, so $I(C)$ is defined. By Lemma 3, every intercalate in C has exactly one of the Types BB, CB, BC. By Lemmas 4–6, the numbers in the three classes are $4\,I(A)\,I(B)$, $m^2\,I(B)$, and $n^2\,I(A)$ respectively. The classes are disjoint because they are distinguished by the equality patterns a and b. Adding the three counts yields

$$I(C) = 4\,I(A)\,I(B) + n^2\,I(A) + m^2\,I(B),$$

for every pair $m \geq 1$, $n \geq 1$ and every pair of Latin squares A, B of those orders.

□

Theorem (c3: Intercalate count of the direct product of Latin squares)**.** *For every pair of positive integers m and n, for every Latin square A of order m and every Latin square B of order n, the direct product Latin square A x B of order m*n satisfies i(A x B) = m^2 * i(B) + n^2 * i(A) + 4 * i(A) * i(B).*

Theorem. *Let A be a Latin square of order m and B a Latin square of order n. Let $A \times B$ be the array with row set $R_A \times R_B$, column set $C_A \times C_B$, and symbol set $S_A \times S_B$ defined by*

$$(A \times B)((r_1, r_2), (c_1, c_2)) = (A(r_1, c_1), B(r_2, c_2)).$$

Then

$$i(A \times B) = m^2\,i(B) + n^2\,i(A) + 4\,i(A)\,i(B),$$

where $i(L)$ is the number of unordered pairs $(\{r, r'\}, \{c, c'\})$ with $r \neq r'$, $c \neq c'$ satisfying $L(r, c) = L(r', c')$ and $L(r, c') = L(r', c)$.

Lemma 1 (Rectangles in a Latin square)**.** *Let L be a Latin square of order N, with rows r, r' and columns c, c'. Then the two equalities*

$$L(r, c) = L(r', c') \quad \textit{and} \quad L(r, c') = L(r', c)$$

both hold if and only if one of the following holds:

- *(i) $r = r'$ and $c = c'$; in this case the four entries are all equal;*
- *(ii) $r \neq r'$ and $c \neq c'$, and $(\{r, r'\}, \{c, c'\})$ is an intercalate of L; in this case $L(r, c) \neq L(r, c')$.*

No other configuration of (r, r', c, c') satisfies both equalities.

Proof of Lemma 1. If (i) holds, both equalities reduce to $L(r, c) = L(r, c)$, which holds, and all four positions coincide, so the values are equal.

If (ii) holds, both equalities hold by the definition of an intercalate. Moreover $L(r, c) \neq L(r, c')$, since these are two entries in the same row r in distinct columns $c \neq c'$, and rows of a Latin square contain no repeated symbol.

Conversely, suppose both equalities hold. If $r = r'$ and $c \neq c'$, then the first equality gives $L(r, c) = L(r, c')$ with $c \neq c'$ in the same row, contradicting the Latin property. Hence $r = r'$ forces $c = c'$, which is case (i). Symmetrically, if $c = c'$ and $r \neq r'$, the first equality gives $L(r, c) = L(r', c)$ with $r \neq r'$ in the same column,

contradicting the Latin property that columns contain no repeated symbol. Hence $c = c'$ forces $r = r'$, again case (i).

The remaining possibility is $r \neq r'$ and $c \neq c'$. Then the two equalities are exactly the intercalate condition for the unordered pairs $\{r, r'\}$ and $\{c, c'\}$. The values on the two diagonals are distinct: if $L(r, c) = L(r, c')$ then row r would repeat a symbol in distinct columns, which is impossible. Since $L(r, c') = L(r', c)$ by hypothesis, this gives $L(r, c) \neq L(r', c)$, i.e. the two symbols of the intercalate are distinct. Thus $(\{r, r'\}, \{c, c'\})$ is an intercalate, which is case (ii). This exhausts all cases.

□

Lemma 2 (The product is Latin)**.** *$A \times B$ is a Latin square of order mn.*

Proof of Lemma 2. The row set, column set, and symbol set each have cardinality mn. Fix a row (r_1, r_2) and two distinct columns $(c_1, c_2) \neq (c_1', c_2')$. Then $c_1 \neq c_1'$ or $c_2 \neq c_2'$ (or both). In the first case, $A(r_1, c_1) \neq A(r_1, c_1')$ because A is Latin, so the ordered pairs $(A(r_1, c_1), B(r_2, c_2))$ and $(A(r_1, c_1'), B(r_2, c_2'))$ differ in the first coordinate. In the second case, $B(r_2, c_2) \neq B(r_2, c_2')$ because B is Latin, so the pairs differ in the second coordinate. Hence every row contains pairwise distinct symbols. Since there are mn columns and mn symbols, each row contains each symbol exactly once. The same argument with rows and columns exchanged shows each column contains each symbol exactly once. Thus $A \times B$ is a Latin square.

□

Lemma 3 (Classification of product intercalates)**.** *Let $R = (r_1, r_2)$, $R' = (r_1', r_2')$ be distinct rows of $A \times B$ and $C = (c_1, c_2)$, $C' = (c_1', c_2')$ distinct columns. Write*

$$a_1 = A(r_1, c_1),\ a_2 = A(r_1, c_1'),\ a_1' = A(r_1', c_1'),\ a_2' = A(r_1', c_1),$$

and analogously b_1, b_2, b_1', b_2' for B. Then $(\{R, R'\}, \{C, C'\})$ is an intercalate of $A \times B$ if and only if

$$a_1 = a_1', \quad a_2 = a_2', \quad b_1 = b_1', \quad b_2 = b_2',$$

with $(a_1, b_1) \neq (a_2, b_2)$, and moreover the A-coordinates satisfy either collapsed or intercalate type and likewise for B-coordinates. Precisely, exactly one of the following three types occurs:

- *Type I: $r_1 = r_1'$, $c_1 = c_1'$ (so $a_1 = a_2$), while $r_2 \neq r_2'$, $c_2 \neq c_2'$ and $(\{r_2, r_2'\}, \{c_2, c_2'\})$ is an intercalate of B (so $b_1 \neq b_2$).*
- *Type II: $r_2 = r_2'$, $c_2 = c_2'$ (so $b_1 = b_2$), while $r_1 \neq r_1'$, $c_1 \neq c_1'$ and $(\{r_1, r_1'\}, \{c_1, c_1'\})$ is an intercalate of A (so $a_1 \neq a_2$).*
- *Type III: $r_1 \neq r_1'$, $c_1 \neq c_1'$ forming an intercalate of A (so $a_1 \neq a_2$), and $r_2 \neq r_2'$, $c_2 \neq c_2'$ forming an intercalate of B (so $b_1 \neq b_2$).*

Proof of Lemma 3. Fix an ordering of the two rows as R, R' and of the two columns as C, C'. The product intercalate condition is

$$(A \times B)(R, C) = (A \times B)(R', C'), \quad (A \times B)(R, C') = (A \times B)(R', C).$$

Equality of ordered pairs holds coordinate-wise, so this is equivalent to the conjunction of the A-equations $a_1 = a_1'$, $a_2 = a_2'$ and the B-equations $b_1 = b_1'$, $b_2 = b_2'$. The additional requirement that the two product symbols differ is $(a_1, b_1) \neq (a_2, b_2)$.

Suppose $(\{R, R'\}, \{C, C'\})$ is an intercalate. Then in particular the A-equations hold. By Lemma 1 applied to A with rows r_1, r_1' and columns c_1, c_1', either $r_1 = r_1'$ and $c_1 = c_1'$ (collapsed A), or $r_1 \neq r_1'$, $c_1 \neq c_1'$ forming an A-intercalate. The mixed cases $r_1 = r_1'$, $c_1 \neq c_1'$ and $r_1 \neq r_1'$, $c_1 = c_1'$ are impossible by Lemma 1, since the equations do hold. The same reasoning applied to B gives either collapsed B ($r_2 = r_2'$, $c_2 = c_2'$) or a B-intercalate ($r_2 \neq r_2'$, $c_2 \neq c_2'$ forming an intercalate).

Both coordinates cannot be collapsed simultaneously: that would give $R = R'$ and $C = C'$, contradicting distinctness of the two rows and two columns. Hence at least one coordinate is of intercalate type, which

already forces $R \neq R'$ and $C \neq C'$. A direct inspection leaves exactly Types I, II, III above. The claims about equality versus inequality of a_1, a_2 and b_1, b_2 follow from Lemma 1: in collapsed type the two values coincide, in intercalate type they differ. In each type at least one of $a_1 \neq a_2$, $b_1 \neq b_2$ holds, so $(a_1, b_1) \neq (a_2, b_2)$ holds.

Conversely, suppose quadruples (r_1, r_1', c_1, c_1') and (r_2, r_2', c_2, c_2') are each of collapsed or intercalate type, not both collapsed. Define R, R', C, C' as above. If the A-type is collapsed then $r_1 = r_1'$, so the two product rows can differ only via $r_2 \neq r_2'$ supplied by the B-intercalate; thus $R \neq R'$. Similarly $C \neq C'$. The same holds with A and B exchanged, and if both are intercalate type then $R \neq R'$ and $C \neq C'$ follow from either coordinate. The coordinate equalities hold by Lemma 1 in each coordinate, hence the product equalities hold, and $(a_1, b_1) \neq (a_2, b_2)$ holds as just shown. Swapping the labels $R \leftrightarrow R'$ or $C \leftrightarrow C'$ preserves these equations, so the unordered pair $(\{R, R'\}, \{C, C'\})$ satisfies the intercalate condition.

□

Lemma 4 (Count of Types I and II)**.** *Let J_1 be the set of intercalates of $A \times B$ of Type I and J_2 those of Type II. Then $|J_1| = m^2\, i(B)$ and $|J_2| = n^2\, i(A)$.*

Proof of Lemma 4. The property defining Type I, namely $r_1 = r_1'$ and $c_1 = c_1'$, depends only on the unordered row pair $\{R, R'\}$ and column pair $\{C, C'\}$, not on the chosen ordering, so J_1 is well defined, and similarly J_2. The three types are mutually exclusive by the equalities versus inequalities among the A- and B-coordinates.

For Type I, define a map

$$\Phi_1 : R_A \times C_A \times \mathcal{I}(B) \to J_1$$

where $\mathcal{I}(B)$ is the set of B-intercalates, by

$$\Phi_1(r_1, c_1, (\{r_2, r_2'\}, \{c_2, c_2'\})) = (\{(r_1, r_2), (r_1, r_2')\}, \{(c_1, c_2), (c_1, c_2')\}).$$

Since the inputs $\{r_2, r_2'\}$ and $\{c_2, c_2'\}$ are unordered sets, the output unordered pairs are independent of how their elements are listed, so Φ_1 is well defined. By Lemma 3 the output is indeed a Type I intercalate.

Φ_1 is injective: from the output, the common first row-coordinate r_1 and common first column-coordinate c_1 are recovered as the shared coordinates of the two product rows and columns, and the B-row-pair and B-column-pair are recovered by projecting to the second coordinates.

Φ_1 is surjective: let $(\{R, R'\}, \{C, C'\}) \in J_1$. By Lemma 3, $R = (r_1, r_2)$, $R' = (r_1, r_2')$ with $r_2 \neq r_2'$, and $C = (c_1, c_2)$, $C' = (c_1, c_2')$ with $c_2 \neq c_2'$, up to ordering within each pair. Projecting to the second coordinates yields $(\{r_2, r_2'\}, \{c_2, c_2'\})$, which is a B-intercalate by Lemma 3, and the element is $\Phi_1(r_1, c_1, \cdot)$ of that B-intercalate. Hence Φ_1 is a bijection. Since $|R_A \times C_A| = m^2$, $|J_1| = m^2\, i(B)$.

The Type II count is symmetric, exchanging the roles of A and B:

$$\Phi_2((\{r_1, r_1'\}, \{c_1, c_1'\}), r_2, c_2) = (\{(r_1, r_2), (r_1', r_2)\}, \{(c_1, c_2), (c_1', c_2)\})$$

is a bijection from $\mathcal{I}(A) \times R_B \times C_B$ onto J_2, giving $|J_2| = n^2\, i(A)$.

□

Lemma 5 (Count of Type III)**.** *Let J_3 be the set of intercalates of $A \times B$ of Type III. Then $|J_3| = 4\, i(A)\, i(B)$.*

Proof of Lemma 5. Let $\mathcal{I}(A)$ and $\mathcal{I}(B)$ be the sets of intercalates of A and B. Consider the projection

$$\pi : J_3 \to \mathcal{I}(A) \times \mathcal{I}(B)$$

sending $(\{R, R'\}, \{C, C'\})$ to the pair of factor intercalates $((\{r_1, r_1'\}, \{c_1, c_1'\}), (\{r_2, r_2'\}, \{c_2, c_2'\}))$ obtained by projecting rows and columns to each coordinate. By Lemma 3 this is well defined on J_3.

Fix (I_A, I_B) with $I_A = (\{x, x'\}, \{y, y'\})$ and $I_B = (\{u, u'\}, \{v, v'\})$. The fibre $\pi^{-1}(I_A, I_B)$ consists of unordered row pairs $\{R, R'\}$ with first-coordinate set $\{x, x'\}$ and second-coordinate set $\{u, u'\}$, together with

unordered column pairs $\{C, C'\}$ with first-coordinate set $\{y, y'\}$ and second-coordinate set $\{v, v'\}$. Because $x \neq x'$ and $u \neq u'$, there are exactly two such row pairs:

$$\{(x,u),(x',u')\} \quad \text{and} \quad \{(x,u'),(x',u)\},$$

and these are distinct as sets: (x,u) belongs to the first but not to the second, since $u \neq u'$ and $x \neq x'$. Likewise there are exactly two such column pairs:

$$\{(y,v),(y',v')\} \quad \text{and} \quad \{(y,v'),(y',v)\},$$

distinct for the same reason. Hence there are $2 \times 2 = 4$ candidate pairs $(\{R, R'\}, \{C, C'\})$ in the fibre.

It remains that each of these 4 candidates is indeed an intercalate. This follows from Lemma 3 and the symmetry of the intercalate condition under reordering: for any choice of orderings, e.g. $R_1 = (x, u^*)$, $R_1' = (x', u^{**})$ with $\{u^*, u^{**}\} = \{u, u'\}$, and $C_1 = (y, v^*)$, $C_1' = (y', v^{**})$ with $\{v^*, v^{**}\} = \{v, v'\}$, the A-equations read $A(x,y) = A(x',y')$ and $A(x,y') = A(x',y)$, which hold because I_A is an intercalate (the condition is invariant under swapping the two row labels or the two column labels, which merely interchanges the two equations). The B-equations read $B(u^*, v^*) = B(u^{**}, v^{**})$ and $B(u^*, v^{**}) = B(u^{**}, v^*)$; these hold because I_B is an intercalate: both the parallel pairing ($B(u,v) = B(u',v')$, $B(u,v') = B(u',v)$) and the crossed pairing ($B(u',v) = B(u,v')$, $B(u',v') = B(u,v)$) are the same two equations. Thus all 4 candidates satisfy the coordinate equalities, have distinct rows and columns, and have $(a_1, b_1) \neq (a_2, b_2)$ since $a_1 \neq a_2$ and $b_1 \neq b_2$. By Lemma 3 they are Type III intercalates.

Therefore every fibre of π has exactly 4 elements, and $|J_3| = 4\,|\mathcal{J}(A)| \cdot |\mathcal{J}(B)| = 4\,i(A)\,i(B)$.

□

Proof of the Theorem. By Lemma 2, $A \times B$ is a Latin square, so $i(A \times B)$ is defined. By Lemma 3, its intercalates partition into the disjoint sets J_1, J_2, J_3 of Types I, II, III. By Lemma 4 and Lemma 5,

$$i(A \times B) = |J_1| + |J_2| + |J_3| = m^2\, i(B) + n^2\, i(A) + 4\, i(A)\, i(B),$$

as required.

□

4. Computational verification

4.1 Exhaustive search ranges

All three formulations were tested by brute-force enumeration of Latin squares, construction of direct products, and independent counting of intercalates by checking all unordered row pairs and column pairs against the formula.

For c1, the search enumerated all 2 Latin squares of order 2 and all 12 Latin squares of order 3, formed every ordered pair (A, B) with orders in $\{2, 3\}$, built the direct product of order mn, counted intercalates, and compared against $n^2 I(A) + m^2 I(B) + 4 I(A) I(B)$. It additionally tested 200 random Latin squares of order 4 in mixed pairs to cover nonzero intercalate factors. The run reported 196 ordered small pairs plus 200 random order-4 mixed pairs, for 396 total cases, with no counterexample. Sanity checks confirmed $I = 1$ for the order-2 square, $I = 0$ for the cyclic order-3 square, enumeration counts 2 and 12, agreement of an alternative counter on 7 squares, and that a 2×3 product is Latin of order 6.

For c2, the search enumerated all Latin squares A of order m in $\{2, 3, 4\}$ (2 of order 2, 12 of order 3, 576 of order 4) and all Latin squares B of order n in $\{2, 3\}$ (2 of order 2, 12 of order 3), formed each direct product C of order mn in $\{4, 6, 8, 9, 12\}$, computed $I(A)$, $I(B)$ and $I(C)$, and verified the equation. This covered order pairs $(2,2)$, $(2,3)$, $(3,2)$, $(3,3)$, $(4,2)$ and $(4,3)$, with 8,260 cases checked and no counterexample. Sanity checks confirmed cyclic and Klein-type examples and agreement of fast and plain counters.

For c3, the search enumerated all Latin squares of orders 2, 3 and 4 completely (2, 12 and 576 squares), formed every pair (A, B) with order-pair in $\{(2,2),(2,3),(3,3),(2,4),(3,4)\}$ exhaustively plus 500 uniformly sampled pairs with orders $(4,4)$, built each direct product of order up to 16, counted intercalates by checking all $\binom{N}{2}^2$ row-pair column-pair combinations, and reported no violation. The run reported $4+24+144+1{,}152+6{,}912$ exhaustive cases plus 500 sampled $(4,4)$ cases, for 8,736 cases, with no counterexample.

4.2 Independent checks

Each formulation had a second independent program that re-derived its lemmas. For c1 it confirmed that the product of two Latin squares is Latin, that distinct rows and columns form an intercalate exactly when both diagonal equalities hold, that degenerate diagonal-equality cases behave as claimed, that $|S| = t^2 + 4I$, and that the product formula holds. For c2 it confirmed labelling invariance, diagonal matching, the BB/CB/BC classification, the $m^2 I(B)$ and $n^2 I(A)$ single-factor counts, the $4I(A)I(B)$ double-intercalate count, and the final sum. For c3 it confirmed the rectangle characterization, that the product is Latin, the product classification, the Type I and II counts, the Type III count, and the final formula. All independent checks reported that all checks passed, with no repairs.

4.3 Lean 4 status and faithfulness

For c1, the Lean 4 file compiles with `sorry`. It contains 6 sorries and depends on `sorryAx`. It is therefore not machine-checked sorry-free. Its back-translation was judged faithful, with no differences and no trivialization. The Lean statement quantifies over m, n with $1 \le m$ and $1 \le n$, over functions A and B with injective rows and columns, defines the direct product componentwise with lexicographic order, defines the intercalate count with $r_1 < r_2$, $s_1 < s_2$, both diagonal equalities and inequality, and concludes the same arithmetic equation.

For c2, the Lean 4 file compiles with `sorry`. It contains 5 sorries and depends on `sorryAx`. It is therefore not machine-checked sorry-free. Its back-translation was judged faithful, with no differences and no trivialization. The Lean statement uses row and column bijectivity for the Latin condition, unordered 2-sets with diagonal equalities and inequality for intercalates, cardinality for counting, and componentwise pairing for the product, with an identical final equation.

For c3, the Lean 4 file compiles with `sorry`. It contains 4 sorries and depends on `sorryAx` and `choice`. It is therefore not machine-checked sorry-free. Its back-translation was judged faithful, with no differences and no trivialization. The Lean statement uses positivity and row and column bijectivity, set cardinality of unordered row-pairs and column-pairs with the two opposite-corner equalities, componentwise product on the product type, and an identical arithmetic conclusion.

A sorry-free Lean proof would certify only the Lean statement as written. The Lean statement must still be read against the informal statement, and the faithfulness verdicts above concern that comparison.

5. Limitations and open questions

No `## GAP` markers remain in the three verified proofs, and the referees reported no issues. The computational evidence is limited to orders 2, 3 and 4 and to sampled $(4,4)$ pairs, with larger orders covered only by proof. The three theorems state the same identity, so they do not give three independent results. Open questions include whether analogous closed forms hold for larger subsquares, for other product or composition constructions, for transversal numbers and autotopism groups mentioned in the seed topic but not proved here, and whether the Lean formalizations can be completed to sorry-free proofs.

5.1 Novelty

Novelty rests on bounded automated searches over arXiv, Crossref and OpenAlex, not on a literature review. Such a search is blind to books, to journals outside those indexes, and to any result phrased differently. None of the searches returned a matching hit, and the verdict in each case was apparently new. The evidence base

in each case is retrieval, not memory-only, so the search was executed and novelty does not rest on model recall alone.

For c1, 32 items were retrieved. The queries run were number of intercalates in the direct product of Latin squares, 2x2 subsquares in composite Latin squares from product constructions, and enumeration of intercalates by counting argument in Latin square products, together with the search terms Latin square direct product intercalates, number of 2x2 subsquares direct product Latin squares, and intercalate enumeration product construction, and the follow-up queries "intercalate" "direct product" Latin square formula and "I(A" "I(B)" Latin squares product subsquares. The closest retrieved items concerned the definition of an intercalate as a 2×2 Latin subsquare and general work on Latin subsquares and full and overlapping products, without an intercalate-counting formula.

For c2, 38 items were retrieved. The queries run were formula for the number of intercalates in the direct product of two Latin squares, enumeration of 2x2 subsquares and intercalates in Latin squares from product constructions, and counting intercalates in a product Latin square by projecting rows, columns and symbols onto the factor squares, together with the search terms Latin square direct product intercalates, intercalates in product construction, and 2x2 subsquares direct product quasigroup, and the follow-up queries "direct product" Latin square intercalate number and "product" Latin square "2x2 subsquare" count. The closest retrieved items concerned random intercalates, maximum intercalates, subsquare existence, and unrelated work, with no stated product formula.

For c3, 35 items were retrieved. The queries run were number of intercalates in direct product of Latin squares, intercalate numbers of Latin squares under product constructions, and counting 2x2 subsquares intercalates in Latin squares combinatorial enumeration, together with the search terms Latin square direct product number of intercalates, intercalates Kronecker product Latin squares, and 2x2 subsquares direct product Latin square enumeration, and the follow-up queries "direct product" Latin square "intercalate" count formula and "i(A" "i(B)" Latin square product subsquares. The closest retrieved items concerned random intercalate asymptotics, Kronecker products for mutually orthogonal Latin squares, and bounds on small subsquares, with no stated enumeration for $A \times B$.

These counts and closest-match descriptions are reported as given by the search records. They do not establish priority and do not exclude prior publication under different terminology or outside the searched indexes.