A Folded Parity Formula for Odd-Order Restricted Involutions

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Abstract

We enumerate involutions of $[2r+1]$ whose arcs have length at least three, recording the number of arcs, crossing pairs, and fixed-point–arc incidences. In characteristic two, with crossing and incidence exponents reduced modulo two, the generating polynomial equals a weighted matching sum on $[r]$. The proof cancels nonfixed reflection orbits and classifies reflection-fixed involutions. Exhaustive computations confirm the identity for $1 \le r \le 6$. The supplied Lean 4 formalization compiles with one `sorry`; it is not a machine-checked proof.

1. Introduction

An involution can be represented by disjoint arcs joining its two-cycle endpoints, with its remaining vertices fixed. Minimum arc-length restrictions connect this representation to restricted partial matchings and diagram enumeration.

The result below combines the restriction $|i-j| \ge 3$ with simultaneous crossing and fixed-point–arc incidence parity. Reflection reduces an odd-order involution to a matching on half its noncentral vertices, yielding an explicit product formula in characteristic two.

The proposed contribution is this particular weighted folding identity, rather than the general reflection-cancellation technique. The supplied bounded search found no matching result, but does not establish originality. Although endpoint-insertion recurrences motivated the seed topic, no such recurrence is proved here.

2. Notation

For a positive integer q, let $[q] = \{1, \dots, q\}$, and set $[0] = \emptyset$. An involution is a permutation π satisfying $\pi^2 = \mathrm{id}$. Represent a two-cycle by an arc (i, j), where $i < j$. Define:

- $m(\pi)$: the number of arcs;
- $c(\pi)$: the number of unordered arc pairs with alternating endpoints;
- $f(\pi)$: the number of pairs consisting of an arc (i, j) and a fixed point k with $i < k < j$.

Work in

$$R = \mathbb{F}_2[t, u, v]/(u^2 - 1, v^2 - 1).$$

Thus crossing and incidence exponents retain only their parity.

A matching M on $[r]$ is a set of pairwise vertex-disjoint two-element subsets. Let

$$a(M) = \#\{i \in [r-1] : i \text{ is incident to no edge of } M\}, \qquad \varepsilon_{ij} = \begin{cases} 1, & |i-j| \ge 3, \\ 0, & |i-j| < 3. \end{cases}$$

Empty products equal 1.

3. Results

Theorem 1 (A folded parity formula for odd-order restricted involutions)**.** *For every integer $r \geq 1$, the following identity holds in R:*

$$\sum_{\pi} t^{m(\pi)} u^{c(\pi)} v^{f(\pi)} = \sum_{M} (1+tv)^{a(M)} \prod_{\{i,j\} \in M} t^2 (u + \varepsilon_{ij}).$$

The left sum ranges over all involutions of $[2r+1]$ whose every two-cycle (i,j) satisfies $|i-j| \geq 3$. The right sum ranges over all matchings on $[r]$, including the empty matching.

The supplied verified proof is reproduced verbatim below.

Proof.

Put $n = 2r+1$, and define the reflection

$$\rho(k) = n + 1 - k = 2r + 2 - k \qquad (k \in [n]).$$

For $i \in [r]$, write $\bar{i} = \rho(i)$. The central vertex is $r+1$.

We first establish the cancellation and parity facts needed in the proof.

Lemma 1 (Reflection cancellation)**.** *Let $\mathcal{I}$ be the set of involutions in the left-hand sum. Reflection acts on $\mathcal{I}$ by*

$$\pi \longmapsto \rho\pi\rho.$$

This action preserves $m(\pi)$, $c(\pi)$, and $f(\pi)$. Consequently, in R,

$$\sum_{\pi \in \mathcal{I}} t^{m(\pi)} u^{c(\pi)} v^{f(\pi)} = \sum_{\substack{\pi \in \mathcal{I} \\ \rho\pi\rho = \pi}} t^{m(\pi)} u^{c(\pi)} v^{f(\pi)}.$$

Proof. Reflection sends an arc (i,j), with $i < j$, to

$$(\rho(j), \rho(i)).$$

Its length remains $j - i$, so the restriction on arc lengths is preserved. Reflection also preserves the number of arcs.

If two arcs have alternating endpoints, their reflected endpoints alternate as well. Thus reflection preserves crossing pairs. Likewise, a fixed point lies strictly between the endpoints of an arc if and only if its reflection lies strictly between the endpoints of the reflected arc. Hence c and f are preserved.

Since reflection has order two, every orbit has one or two elements. A two-element orbit contributes twice the same monomial, which is zero because R has characteristic two. Only the one-element orbits remain.

□

Lemma 2 (Structure of reflection-fixed involutions)**.** *Reflection-fixed members of $\mathcal{I}$ correspond bijectively to the following data:*

1. *a matching M on $[r]$;*
2. *for every vertex i unused by M, a choice between two fixed points $i, \bar{i}$ and the symmetric arc $(i, \bar{i})$, with the latter choice permitted exactly when $i \leq r-1$;*
3. *for every edge $\{i,j\} \in M$, with $i < j$, a choice of the crossing lift*

$$(i, \bar{j}), \quad (j, \bar{i}),$$

or the same-side lift

$$(i, j), \quad (\bar{j}, \bar{i}),$$

with the same-side lift permitted exactly when $j - i \geq 3$.

The central vertex is fixed, and the crossing lift is always permitted.

Proof. A reflection-fixed involution π commutes with ρ. Therefore $\pi(r+1)$ is fixed by ρ, and, since $r+1$ is the unique such vertex,

$$\pi(r+1) = r+1.$$

For each $i \in [r]$, consider the mirrored pair $P_i = \{i, \bar{i}\}$. If one vertex of P_i is fixed, commutation with reflection forces both to be fixed. If a vertex of P_i is paired with the other vertex of P_i, the resulting arc is $(i, \bar{i})$. Otherwise, it is paired with a vertex of some P_j, $j \neq i$; reflection then forces the other two vertices to be paired. These four vertices yield exactly one of the two displayed lifts.

Each mirrored pair can participate in at most one such connection, so the connections form a matching M. These alternatives exhaust the possibilities and recover all the stated data uniquely.

The symmetric arc has length

$$\bar{i} - i = 2(r+1-i),$$

which is at least three exactly when $i \leq r-1$. The two same-side arcs each have length $j-i$, giving their stated restriction. The crossing arcs each have length

$$2r+2-i-j.$$

Since distinct $i, j \in [r]$ satisfy $i+j \leq 2r-1$, this length is at least three.

Conversely, any stated choices use disjoint vertices, define an involution commuting with reflection, and satisfy every arc-length restriction. This proves the bijection.

□

Lemma 3 (Fixed-point incidence parity)**.** *For any involution with arc set A,*

$$f(\pi) \equiv \sum_{(i,j)\in A} (j-i-1) \pmod 2.$$

For a reflection-fixed involution, this implies

$$f(\pi) \equiv s \pmod 2,$$

where s is its number of symmetric arcs.

Proof. For an arc (i, j), its $j-i-1$ interior vertices consist of fixed points and endpoints of other arcs. Summing over all arcs gives

$$\sum_{(i,j)\in A} (j-i-1) = f(\pi) + E,$$

where E counts incidences of an arc with an interior endpoint of another arc.

For each unordered pair of arcs, disjoint intervals contribute zero to E, nested intervals contribute two, and alternating endpoints contribute two. These are all possibilities because arc endpoints are distinct. Thus E is even, proving the first assertion.

In a reflection-fixed involution, every nonsymmetric arc occurs with a distinct reflected arc of equal length. Their contributions to the displayed sum are equal and therefore have even sum. A symmetric arc has even length, so its contribution $j-i-1$ is odd. The second assertion follows.

□

Lemma 4 (Crossing parity)**.** *For a reflection-fixed involution,*

$$c(\pi) \equiv x \pmod 2,$$

where x is the number of edges whose chosen lift is the crossing lift.

Proof. Reflection acts on the set of unordered crossing pairs of arcs. Orbits of size two make an even contribution to $c(\pi)$. A reflection-fixed unordered pair of distinct arcs must either consist of two individually reflection-fixed arcs or consist of two arcs exchanged by reflection.

An individually reflection-fixed arc is symmetric. If $i < j$, the symmetric arcs have endpoints ordered as

$$i < j < \bar{j} < \bar{i},$$

so they are nested, not crossing. Thus the first possibility produces no fixed crossing pair.

In the second possibility, the two arcs form one of the lifts in Lemma 2. For a crossing lift, the endpoint order

$$i < j < \bar{j} < \bar{i}$$

shows that $(i, \bar{j})$ and $(j, \bar{i})$ cross. For a same-side lift, the two arcs occupy disjoint intervals and do not cross. Hence the fixed crossing pairs are exactly those arising from crossing lifts, proving the claimed parity.

□

Completion of the proof

By Lemma 1, it suffices to sum over reflection-fixed involutions. Fix a matching M, and use the choices of Lemma 2. If s symmetric arcs and x crossing lifts are chosen, then

$$m(\pi) = s + 2|M|.$$

Lemmas 3 and 4, together with $u^2 = v^2 = 1$, give

$$t^{m(\pi)} u^{c(\pi)} v^{f(\pi)} = t^{s+2|M|} u^x v^s.$$

Each unused vertex in $[r-1]$ therefore contributes a choice of weight 1 or tv. The unused vertex r, if present, allows only two fixed points and contributes 1. These choices give

$$(1 + tv)^{a(M)}.$$

For an edge $\{i, j\}$, the crossing lift contributes $t^2 u$. The same-side lift contributes t^2 when $|i - j| \geq 3$, and is unavailable otherwise. Thus its total contribution is

$$t^2(u + \varepsilon_{ij}).$$

The choices are independent by the bijection in Lemma 2, and their weights multiply by the parity formulas. Summing first over choices for each M, and then over all matchings, yields

$$\sum_{\pi} t^{m(\pi)} u^{c(\pi)} v^{f(\pi)} = \sum_{M} (1 + tv)^{a(M)} \prod_{\{i,j\} \in M} t^2(u + \varepsilon_{ij}),$$

as asserted. The argument also applies when $r = 1$, with $[r - 1] = \emptyset$ and the stipulated empty-product conventions. ■

4. Computational verification

The following summarizes supplied verification records; no new computation is claimed here.

Exhaustive search

For each $1 \le r \le 6$, the search enumerated involutions of $[2r+1]$, rejected arcs shorter than three, and tabulated

$$(m, c \bmod 2, f \bmod 2)$$

modulo two. It separately enumerated matchings on $[r]$, expanded the right-hand side in R, and compared every coefficient

$$[t^m u^\alpha v^\beta], \qquad 0 \le m \le r, \quad \alpha, \beta \in \{0, 1\}.$$

The reported search found no counterexample: six identities, 108 coefficients, 119,470 admissible involutions, and 119 matchings were tested. Sanity checks included the 76 unrestricted involutions on $[6]$, the polynomial $1 + t$ for admissible involutions on $[4]$, a permutation brute-force check on $[5]$, and an independent right-hand-side expansion for $r = 1, \dots, 4$.

Independent check

A separate supplied check reported these admissible counts:

r	$2r+1$	Admissible involutions
1	3	1
2	5	5
3	7	47
4	9	532
5	11	7,136
6	13	111,749

It confirmed reflection invariance and cancellation, the folded classification and length restrictions, both incidence-parity statements, crossing parity computed directly from arcs, individual weights, fixed-matching factorization, and equality of independently enumerated sides. The unrestricted incidence-parity check covered 13,231 cases; its size range was not specified in the supplied output. All checks passed.

The supplied referee assessment also marked the proof valid, with no issues, including $r = 1$.

Lean 4 status

Theorem	Lean 4 status	Back-translation faithfulness
Theorem 1	**Compiles with `sorry`**	**FAITHFUL**, according to the supplied assessment

The compilation exited successfully but reported one `sorry`. The main theorem depends on `sorryAx`, alongside `propext`, `Classical.choice`, and `Quot.sound`; therefore it is **not machine-checked sorry-free**.

The back-translation uses zero-based vertex sets $\{0, \dots, 2r\}$ and $\{0, \dots, r-1\}$. Shifting indices by one preserves order, distances, statistics, and the unmatched-vertex condition. The supplied assessment reports no differences or trivialization.

A sorry-free Lean proof certifies the **Lean statement**, which still has to be read against the informal statement. Here, the faithful-statement assessment does not supply the missing proof.

5. Limitations and open questions

The theorem retains crossing and incidence parity, not their full integer distributions. Finite computations support the result but cannot establish it for all r; that conclusion rests on the combinatorial proof.

No `## GAP` markers occur in the supplied verified proof. The remaining formal-verification gap is explicit: the Lean theorem contains one `sorry`. Removing it is an open verification task.

Further questions include even-order folding formulas, other minimum arc lengths, and endpoint-insertion recurrences tracking open arcs and all three unreduced statistics. None is established here.

Novelty

The supplied novelty record labels the result "apparently new," with `evidence_base: retrieval`, not `memory-only`. It reports one search round using the following exact queries against arXiv, Crossref, and OpenAlex:

```
("restricted involutions" OR "partial matchings") ("minimum arc length 3" OR "minimum arc length three"
```

```
("RNA secondary structures" OR "RNA diagrams" OR "partial matchings") "minimum arc length" "crossing nu
```

```
("sign-reversing involution" OR "involutive cancellation" OR "Pfaffian identity") ("matching polynomial
```

The supplied retrieved list contains **29 entries: 15 arXiv and 14 Crossref entries**. Its apparent query grouping is:

Query	arXiv entries	Crossref entries	OpenAlex
First	5	5	HTTP 429; no results retrieved
Second	5	4	HTTP 429; no results retrieved
Third	5	5	HTTP 429; no results retrieved

These are counts of entries supplied in the record, not reported total index hit counts; total hit counts were not provided. All three OpenAlex requests failed with "Too Many Requests." The record lists no matching hits. Retrieved related titles include *Restricted involutions and Motzkin paths*, *Transformations of partial matchings*, and *A sign-reversing involution for an extension of Torelli's Pfaffian identity*. Their retrieval alone does not show that their full contents were reviewed.

This was a **bounded automated search over those indexes—not a literature review**. It is blind to books, journals outside those indexes, and anything phrased differently. The OpenAlex failures further limit coverage. Consequently, originality remains provisional. No `memory-only` evidence base is present here; if it were, the search would have failed and novelty would rest on model recall alone.

Models

A folded parity formula for odd-order restricted involutions (`verified`)

- Proposed the claim: gpt-6.1-sol
- Counterexample search: gpt-6.1-sol
- Wrote the proof: gpt-6.1-sol
- Referee: gpt-6.1-sol
- Independent check script: gpt-6.1-sol
- Lean formalization: gpt-6.1-sol
- Faithfulness judge: gpt-6.1-sol