

Torus Divisibility for Refined Unitriangular Class Counts

Augusto Rehfeldt

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Abstract

This paper records two verified divisibility statements about conjugacy classes of the unitriangular groups $UT_n(\mathbf{F}_p)$ over small prime fields. Both state that class counts refined by exact superdiagonal support S , Jordan block partition λ , and in one case centralizer order p^e , are multiples of $(p-1)^{|S|}$. The proofs use only that UT_n -conjugation fixes each superdiagonal entry while diagonal torus conjugation normalizes UT_n , preserves the refined invariants, and acts transitively on value patterns with fixed support. Both statements were tested by exhaustive enumeration over small n and p with no counterexample found, and were independently re-checked. Neither has a sorry-free Lean 4 formalization. Novelty claims rest only on a bounded automated search described in Section 5.

1. Introduction

Context

Let $UT_n(\mathbf{F}_p)$ be the group of $n \times n$ upper-triangular matrices over $\mathbf{F}_p$ with ones on the diagonal. Its conjugacy classes, centralizer sizes, commuting tuples, and class numbers $k(UT_n(\mathbf{F}_q))$ are classical subjects, including Higman's conjecture on polynomiality in q , supercharacter theories, coadjoint orbits, and the normalizing action of the diagonal torus.

The seed topic for this work was conjugacy classes, centralizer orders, and commuting pairs and triples in $UT_n(\mathbf{F}_p)$, refined by Jordan form, superdiagonal support patterns, and commutator membership, looking for class-number formulas, divisibility conditions, and branching identities.

What is new

Two results survived verification:

1. A triple refinement: the number $N(n, p, S, \lambda, e)$ of $UT_n(\mathbf{F}_p)$ -classes with exact support S , Jordan type λ , and centralizer order p^e is divisible by $(p-1)^{|S|}$.
2. A double refinement: the number $N_{n,p}(\lambda, S)$ of classes with Jordan type λ and support S is divisible by $(p-1)^{|S|}$.

The method is standard: UT_n -invariance of superdiagonal entries plus torus transitivity over fixed support. The specific jointly refined divisibility statements were not found in the bounded search described in Section 5, and are therefore reported as apparently new subject to the limitations stated there.

2. Notation

$UT_n(\mathbf{F}_p)$ is the group of $n \times n$ upper-triangular matrices over the field with p elements with all diagonal entries equal to 1.

For $A \in UT_n(\mathbf{F}_p)$, $S(A) = \{i \in \{1, \dots, n-1\} : A_{i,i+1} \neq 0\}$ and $|S|$ is its cardinality.

$\lambda(A)$ is the partition of n given by the sizes of the Jordan blocks of A , which all have eigenvalue 1 since A is unipotent, equivalently of the nilpotent $A - I$.

Centralizer order is $|\{G \in UT_n(\mathbf{F}_p) : GA = AG\}|$, which equals p^e for some $e \geq 0$.

$N(n, p, S, \lambda, e)$ counts $UT_n(\mathbf{F}_p)$ -conjugacy classes with these three invariants equal to the prescribed S, λ, e , with $N = 0$ if no such class exists and 0 deemed divisible.

$N_{n,p}(\lambda, S)$ counts $UT_n(\mathbf{F}_p)$ -conjugacy classes C such that every $g \in C$ has Jordan type λ and $S(g) = S$, with $(p-1)^0 = 1$.

$\mathcal{D}$ denotes diagonal matrices with nonzero diagonal entries. V_S denotes vectors in $\mathbf{F}_p^{n-1}$ with nonzero entries exactly on S . $v(g) = (g_{12}, \dots, g_{n-1,n})$.

3. Results

Theorem 1 (Torus divisibility for refined unitriangular class counts). *For every integer $n \geq 2$, every prime p , every subset S of $\{1, \dots, n-1\}$, every partition λ of n , and every integer $e \geq 0$, let $N(n, p, S, \lambda, e)$ be the number of conjugacy classes C of $UT_n(\mathbf{F}_p)$ such that every matrix A in C has superdiagonal support exactly S , Jordan type exactly λ , and centralizer order in $UT_n(\mathbf{F}_p)$ exactly p^e ; then $(p-1)^{|S|}$ divides $N(n, p, S, \lambda, e)$.*

Full proof, verbatim from the verified proof:

Theorem. *Let $n \geq 2$ be an integer, p a prime, $S \subseteq \{1, \dots, n-1\}$, λ a partition of n , and $e \geq 0$ an integer. Let $UT_n(\mathbf{F}_p)$ denote the group of $n \times n$ upper-triangular matrices over $\mathbf{F}_p$ with all diagonal entries equal to 1. For $A \in UT_n(\mathbf{F}_p)$ let*

$$S(A) = \{i \in \{1, \dots, n-1\} : A_{i,i+1} \neq 0\},$$

let $\lambda(A)$ be the partition of n given by the sizes of the Jordan blocks of A over an algebraic closure of $\mathbf{F}_p$ (all with eigenvalue 1), and let $|C_{UT}(A)|$ denote the order of

$$C_{UT}(A) = \{G \in UT_n(\mathbf{F}_p) : GA = AG\}.$$

Let $N(n, p, S, \lambda, e)$ be the number of $UT_n(\mathbf{F}_p)$ -conjugacy classes C such that every $A \in C$ satisfies $S(A) = S$, $\lambda(A) = \lambda$, and $|C_{UT}(A)| = p^e$ (with the count 0 if no such class exists). Then $(p-1)^{|S|}$ divides $N(n, p, S, \lambda, e)$.

Lemmas

Lemma 1. *$UT_n(\mathbf{F}_p)$ is a finite subgroup of $GL_n(\mathbf{F}_p)$ of order $p^{n(n-1)/2}$, and the inverse of a unitriangular matrix is unitriangular.*

Proof. A matrix is in $UT_n(\mathbf{F}_p)$ if and only if its diagonal entries are 1, its below-diagonal entries are 0, and its $n(n-1)/2$ strictly above-diagonal entries are arbitrary elements of $\mathbf{F}_p$. Hence as a set it has cardinality $p^{n(n-1)/2}$ and contains the identity. Its determinant is the product of diagonal entries, hence 1, so it is a subset of $GL_n(\mathbf{F}_p)$.

Let A, B be unit upper-triangular. For $i > j$,

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}.$$

If $k < i$ then $A_{ik} = 0$ because A is upper-triangular. If $k \geq i$ then $k \geq i > j$, so $k > j$ and $B_{kj} = 0$ because B is upper-triangular. Thus every term is 0 and $(AB)_{ij} = 0$. For the diagonal,

$$(AB)_{ii} = \sum_k A_{ik} B_{ki}.$$

If $k < i$ then $A_{ik} = 0$; if $k > i$ then $B_{ki} = 0$; the only surviving term is $k = i$, giving $A_{ii}B_{ii} = 1 \cdot 1 = 1$. Hence AB is unit upper-triangular. So the set is finite, nonempty, and closed under multiplication inside the group $GL_n(\mathbb{F}_p)$. A finite nonempty subset of a group closed under multiplication is a subgroup: for any A in the set, the powers $A, A^2, \dots$ cannot all be distinct, so $A^r = A^s$ for some $r > s \geq 1$, giving $A^{r-s} = I$ and $A^{-1} = A^{r-s-1}$ in the set. Therefore inverses remain unit upper-triangular and $UT_n(\mathbb{F}_p)$ is a subgroup of the stated order. $\square$

Lemma 2. *Let $D = \text{diag}(d_1, \dots, d_n)$ with $d_i \in \mathbb{F}_p^*$, and let A be any $n \times n$ matrix over $\mathbb{F}_p$. Then D is invertible with $D^{-1} = \text{diag}(d_1^{-1}, \dots, d_n^{-1})$ and*

$$(DAD^{-1})_{ij} = d_i A_{ij} d_j^{-1} \quad \text{for all } i, j.$$

Proof. Left multiplication by a diagonal matrix scales row i by d_i , so $(DA)_{ij} = d_i A_{ij}$. Right multiplication by a diagonal matrix scales column j by d_j^{-1} , so $((DA)D^{-1})_{ij} = (DA)_{ij} d_j^{-1} = d_i A_{ij} d_j^{-1}$. $\square$

Lemma 3. *Let $\mathcal{D}$ be the group of diagonal matrices with nonzero diagonal entries in $\mathbb{F}_p$. Then $\mathcal{D}$ normalizes $UT_n(\mathbb{F}_p)$: for $D \in \mathcal{D}$ and $U \in UT_n(\mathbb{F}_p)$, $DUD^{-1} \in UT_n(\mathbb{F}_p)$. Conjugation by D is an automorphism of $UT_n(\mathbb{F}_p)$.*

Proof. Write $D = \text{diag}(d_1, \dots, d_n)$ and $U \in UT_n$. By Lemma 2, $(DUD^{-1})_{ij} = d_i U_{ij} d_j^{-1}$. If $i > j$ then $U_{ij} = 0$, so the entry remains 0. If $i = j$ then $d_i \cdot 1 \cdot d_i^{-1} = 1$. Above the diagonal the entries are in $\mathbb{F}_p$. Hence $DUD^{-1} \in UT_n(\mathbb{F}_p)$ by the description in Lemma 1. The same holds for $D^{-1} \in \mathcal{D}$, which gives the inverse map. Conjugation is a group homomorphism, hence an automorphism of $UT_n(\mathbb{F}_p)$. $\square$

Lemma 4. *Conjugation by $UT_n(\mathbb{F}_p)$ preserves each superdiagonal entry: for $U, A \in UT_n(\mathbb{F}_p)$ and $B = UAU^{-1}$, $B_{i,i+1} = A_{i,i+1}$ for every $i = 1, \dots, n-1$.*

Proof. By Lemma 1, U, A, U^{-1} are upper-triangular with all diagonal entries 1. For any two upper-triangular matrices the product is upper-triangular, so UA is upper-triangular with diagonal entries 1. Compute

$$(UA)_{i,i+1} = \sum_k U_{ik} A_{k,i+1}.$$

$U_{ik} = 0$ unless $k \geq i$, and $A_{k,i+1} = 0$ unless $k \leq i+1$, so only $k = i, i+1$ contribute. Using $U_{ii} = 1$ and $A_{i+1,i+1} = 1$,

$$(UA)_{i,i+1} = A_{i,i+1} + U_{i,i+1}.$$

Next,

$$B_{i,i+1} = ((UA)U^{-1})_{i,i+1} = \sum_k (UA)_{ik} (U^{-1})_{k,i+1}.$$

Again $(UA)_{ik} = 0$ unless $k \geq i$ and $(U^{-1})_{k,i+1} = 0$ unless $k \leq i+1$, so only $k = i, i+1$ contribute. Using diagonal entries 1,

$$B_{i,i+1} = (UA)_{ii} (U^{-1})_{i,i+1} + (UA)_{i,i+1} (U^{-1})_{i+1,i+1} = (U^{-1})_{i,i+1} + (UA)_{i,i+1}.$$

Substituting the previous identity, $B_{i,i+1} = (U^{-1})_{i,i+1} + A_{i,i+1} + U_{i,i+1}$. Moreover $(UU^{-1})_{i,i+1} = 0$ since $i \neq i+1$, while expanding gives $U_{ii} (U^{-1})_{i,i+1} + U_{i,i+1} (U^{-1})_{i+1,i+1} = (U^{-1})_{i,i+1} + U_{i,i+1}$. Hence $(U^{-1})_{i,i+1} + U_{i,i+1} = 0$, and $B_{i,i+1} = A_{i,i+1}$. $\square$

Lemma 5. *If $A, B \in UT_n(\mathbb{F}_p)$ are UT_n -conjugate, i.e. $B = UAU^{-1}$ for some $U \in UT_n(\mathbb{F}_p)$, then $S(B) = S(A)$, $\lambda(B) = \lambda(A)$, and $|C_{UT}(B)| = |C_{UT}(A)|$.*

Proof. $S(B) = S(A)$ follows from Lemma 4, since each superdiagonal entry is equal, so in particular its vanishing or nonvanishing is the same.

For Jordan type, write $N = A - I$ and $M = B - I$. Then N and M are nilpotent and $M = UNU^{-1}$. By induction $M^k = UN^kU^{-1}$ for every $k \geq 0$. Hence $x \mapsto Ux$ is a linear isomorphism of $\mathbb{F}_p^n$ carrying $\ker N^k$ onto $\ker M^k$, so $\dim_{\mathbb{F}_p} \ker(A - I)^k = \dim_{\mathbb{F}_p} \ker(B - I)^k$ for all k . Rank is determined by the largest nonvanishing minor, so rank and hence nullity of a matrix over $\mathbb{F}_p$ are unchanged upon extension of scalars to an algebraic closure $\overline{\mathbb{F}_p}$. The Jordan form of the unipotent A over $\overline{\mathbb{F}_p}$ consists of blocks for eigenvalue 1 of sizes forming $\lambda(A)$, equivalently the nilpotent N is a direct sum of nilpotent Jordan blocks of those sizes. For one nilpotent Jordan block of size m , $\dim \ker J_m(0)^k = \min(k, m)$. For a direct sum, dimensions add, so with $d_k = \dim \ker N^k$,

$$d_k = \sum_j \min(k, \lambda(A)_j).$$

Writing m_t for the number of parts $\geq t$, this equals $\sum_{t=1}^k m_t$, so $m_k = d_k - d_{k-1}$. Thus the sequence $(d_k)_{k \geq 0}$ determines all m_k and hence the partition. Since the sequences for A and B coincide, $\lambda(A) = \lambda(B)$.

For centralizers, $G \in UT_n$ satisfies $GA = AG$ if and only if $(UGU^{-1})(UBU^{-1}\text{-conjugate})$ commutes with B , because $(UGU^{-1})B = UGAU^{-1}$ and $B(UGU^{-1}) = UAGU^{-1}$, which are equal if and only if $GA = AG$. Thus $G \mapsto UGU^{-1}$ is a bijection of UT_n , by Lemma 1, restricting to a bijection $C_{UT}(A) \rightarrow C_{UT}(B)$. Hence the two centralizers have the same order. □

Lemma 6. *Let $D \in \mathcal{D}$ and $A \in UT_n(\mathbb{F}_p)$, and put $A^D = DAD^{-1}$. Then $A^D \in UT_n(\mathbb{F}_p)$, $S(A^D) = S(A)$, $\lambda(A^D) = \lambda(A)$, and $|C_{UT}(A^D)| = |C_{UT}(A)|$.*

Proof. $A^D \in UT_n$ by Lemma 3. By Lemma 2, $(A^D)_{i,i+1} = d_i d_{i+1}^{-1} A_{i,i+1}$ with $d_i d_{i+1}^{-1} \neq 0$. Hence $(A^D)_{i,i+1} = 0$ if and only if $A_{i,i+1} = 0$, so $S(A^D) = S(A)$. Since $D \in GL_n(\mathbb{F}_p)$, A^D is similar to A ; the same nullity argument as in Lemma 5 applied with U replaced by D gives $\dim \ker(A^D - I)^k = \dim \ker(A - I)^k$ for all k , hence the same Jordan partition. Finally, $G \in UT_n$ commutes with A if and only if DGD^{-1} commutes with A^D , since $(DGD^{-1})(DAD^{-1}) = DGAD^{-1}$ and $(DAD^{-1})(DGD^{-1}) = DAGD^{-1}$. By Lemma 3, $G \mapsto DGD^{-1}$ is a bijection of UT_n restricting to a bijection $C_{UT}(A) \rightarrow C_{UT}(A^D)$, so the orders are equal. □

Lemma 7. *Conjugation by $D \in \mathcal{D}$ sends UT_n -conjugacy classes to UT_n -conjugacy classes: if C is the UT_n -class of A , then DCD^{-1} is the UT_n -class of DAD^{-1} .*

Proof. Write $C = \{UAU^{-1} : U \in UT_n\}$. Then

$$DCD^{-1} = \{DUAU^{-1}D^{-1} : U \in UT_n\} = \{(DUD^{-1})(DAD^{-1})(DUD^{-1})^{-1} : U \in UT_n\}.$$

By Lemma 3, as U ranges over UT_n , $V = DUD^{-1}$ ranges bijectively over UT_n . Hence the set equals $\{V(DAD^{-1})V^{-1} : V \in UT_n\}$, the UT_n -class of DAD^{-1} . □

Lemma 8. *Let $S \subseteq \{1, \dots, n-1\}$ and let*

$$V_S = \{v \in \mathbb{F}_p^{n-1} : v_i \neq 0 \iff i \in S\}.$$

Then $|V_S| = (p-1)^{|S|}$. Moreover, for any $v, w \in V_S$ there exists $D = \text{diag}(d_1, \dots, d_n) \in \mathcal{D}$ such that for every $A \in UT_n$ with superdiagonal vector v , DAD^{-1} has superdiagonal vector w ; D^{-1} performs the reverse transformation.

Proof. Specifying $v \in V_S$ amounts to choosing a nonzero value in $\mathbb{F}_p^*$ for each $i \in S$ and 0 elsewhere, so $|V_S| = (p-1)^{|S|}$, with the convention $(p-1)^0 = 1$ when $S = \emptyset$.

Given $v, w \in V_S$, define $t_i = w_i v_i^{-1} \in \mathbb{F}_p^*$ for $i \in S$ and $t_i = 1$ for $i \notin S$. Define $d_n = 1$ and $d_i = \prod_{k=i}^{n-1} t_k$ for $i < n$. Each $d_i \in \mathbb{F}_p^*$ and $d_i d_{i+1}^{-1} = t_i$ for all i . Let $D = \text{diag}(d_1, \dots, d_n)$. By Lemma 2, for A with $A_{i,i+1} = v_i$, $(DAD^{-1})_{i,i+1} = t_i v_i$, which equals w_i if $i \in S$ and $1 \cdot 0 = 0 = w_i$ if $i \notin S$. Thus DAD^{-1} has vector w . Since $D^{-1} = \text{diag}(d_1^{-1}, \dots, d_n^{-1})$ has consecutive ratios t_i^{-1} , the same computation gives $(D^{-1}BD^{-1})$ vector v for any B with vector w , because $t_i^{-1} w_i = v_i$ on S . □

Proof of the theorem. Fix n, p, S, λ, e . $UT_n(\mathbb{F}_p)$ is finite, so there are finitely many conjugacy classes. For $v \in V_S$ as in Lemma 8, define

$$X_v = \{A \in UT_n(\mathbb{F}_p) : A_{i,i+1} = v_i \ \forall i, \ \lambda(A) = \lambda, \ |C_{UT}(A)| = p^e\}.$$

Let $X = \bigsqcup_{v \in V_S} X_v$, the set of matrices with support exactly S (since v ranges over all vectors with support S), Jordan type λ , and centralizer order p^e .

By Lemma 4 and Lemma 5, UT_n -conjugation preserves the exact superdiagonal vector and the other two invariants. Hence UT_n acts on X by conjugation preserving each X_v : if $A \in X_v$ and $U \in UT_n$, then $UAU^{-1} \in X_v$. Consequently every UT_n -conjugacy class meeting X is contained in a unique X_v , and every UT_n -conjugacy class contained in some X_v satisfies the defining property that all its members have support S , Jordan type λ , and centralizer order p^e .

Let N_v be the number of UT_n -conjugacy classes contained in X_v . Then

$$N(n, p, S, \lambda, e) = \sum_{v \in V_S} N_v,$$

with the sum equal to 0 if all X_v are empty.

Fix $v, w \in V_S$ and choose $D \in \mathcal{D}$ as in Lemma 8, so $A \mapsto DAD^{-1}$ maps vectors v to w and D^{-1} maps w to v . By Lemma 6, this map restricts to a bijection $X_v \rightarrow X_w$: Jordan type and centralizer order are preserved, and the superdiagonal vector is sent from v to w . By Lemma 7, $C \mapsto DCD^{-1}$ maps UT_n -classes to UT_n -classes, and it maps classes contained in X_v to classes contained in X_w , since representatives are sent from X_v to X_w . The map $E \mapsto D^{-1}ED^{-1}$ with D^{-1} in place of D is its two-sided inverse on classes. Hence it is a bijection between the collections of UT_n -classes in X_v and in X_w . Therefore $N_v = N_w$.

Thus all N_v equal a common nonnegative integer m . Summing over V_S and using $|V_S| = (p-1)^{|S|}$ from Lemma 8,

$$N(n, p, S, \lambda, e) = |V_S| \cdot m = (p-1)^{|S|} \cdot m.$$

Since $(p-1)^{|S|} \geq 1$, this exhibits $N(n, p, S, \lambda, e)$ as an integer multiple of $(p-1)^{|S|}$, including the case $m = 0$ where the count is 0. This is the claimed divisibility. □

Theorem 2 (Torus scaling divisibility for joint Jordan-support class counts in $UT_n(\mathbf{F}_p)$). *For every integer $n \geq 1$, every prime p , every partition λ of n , and every subset S of $\{1, \dots, n-1\}$, the number $N_{\{n,p\}}(\lambda, S)$ of conjugacy classes of $UT_n(\mathbf{F}_p)$ under conjugation by $UT_n(\mathbf{F}_p)$ whose elements have Jordan block partition λ and superdiagonal support exactly S is divisible by $(p-1)^{|S|}$.*

Full proof, verbatim from the verified proof:

Theorem. *Let $n \geq 1$ be an integer, p a prime, λ a partition of n , and $S \subset \{1, \dots, n-1\}$. Let $U = UT_n(\mathbf{F}_p)$ be the group of $n \times n$ upper triangular matrices over $\mathbf{F}_p$ with ones on the diagonal. For $g \in U$, let $S(g) = \{i : g_{i,i+1} \neq 0\}$ and let $\lambda(g)$ be the Jordan block partition of g . Let $N_{n,p}(\lambda, S)$ be the number of conjugacy classes C of U under conjugation by U such that every $g \in C$ has $\lambda(g) = \lambda$ and $S(g) = S$. Then $(p-1)^{|S|}$ divides $N_{n,p}(\lambda, S)$, with $(p-1)^0 = 1$.*

Notation

Write $v(g) = (g_{12}, g_{23}, \dots, g_{n-1,n}) \in \mathbf{F}_p^{n-1}$. For $n = 1$, $v(g)$ is the empty tuple. For $S \subset \{1, \dots, n-1\}$, define

$$V_S = \{v \in \mathbf{F}_p^{n-1} : v_i \neq 0 \iff i \in S\}.$$

Thus $S(g) = S$ iff $v(g) \in V_S$. Let $D = \{\text{diag}(d_1, \dots, d_n) : d_i \in \mathbf{F}_p^\times\}$. For $d \in D$ and $v \in \mathbf{F}_p^{n-1}$, define $(d \cdot v)_i = d_i d_{i+1}^{-1} v_i$.

Lemmas

Lemma 1. *Let $k \geq 2$ and let $A^{(1)}, \dots, A^{(k)}$ be $n \times n$ strictly upper triangular matrices over a field. Then the product $A^{(1)} \dots A^{(k)}$ has zero entries on the diagonal and on the first superdiagonal.*

Proof. Write $A_{a,b}^{(t)} = 0$ whenever $a \geq b$, by strict upper triangularity. For indices i, j ,

$$(A^{(1)} \dots A^{(k)})_{ij} = \sum_{m_1, \dots, m_{k-1}} A_{i, m_1}^{(1)} A_{m_1, m_2}^{(2)} \dots A_{m_{k-1}, j}^{(k)}.$$

A summand can be nonzero only if $i < m_1 < m_2 < \dots < m_{k-1} < j$. Take $j = i + 1$. Then a nonzero summand would require $i < m_1 < i + 1$. Since m_1 is an integer, no such m_1 exists. Hence every summand is zero, and the $(i, i + 1)$ entry is zero. The same argument with $j = i$ gives that the diagonal entry is zero, since $i < m_1$ and $m_{k-1} < i$ cannot both hold. $\square$

Lemma 2. *For any $x, g \in U$, with $h = xgx^{-1}$, one has $h_{i,i+1} = g_{i,i+1}$ for all $1 \leq i < n$. In particular, $v(h) = v(g)$ and $S(h) = S(g)$.*

Proof. Since U is a group, $x^{-1} \in U$. Write $x = I + X$, $g = I + G$, $x^{-1} = I + \bar{X}$, where $X, G, \bar{X}$ are strictly upper triangular. From $xx^{-1} = I$,

$$(I + X)(I + \bar{X}) = I,$$

so $X + \bar{X} + X\bar{X} = 0$. By Lemma 1, $X\bar{X}$ vanishes on the diagonal and superdiagonal, so $X + \bar{X}$ vanishes there as well. Moreover,

$$\begin{aligned} h &= (I + X)(I + G)(I + \bar{X}) \\ &= I + X + G + \bar{X} + XG + X\bar{X} + G\bar{X} + XG\bar{X}. \end{aligned}$$

Each of XG , $X\bar{X}$, $G\bar{X}$, $XG\bar{X}$ is a product of at least two strictly upper triangular matrices, so by Lemma 1 each vanishes on the superdiagonal. Hence for each i ,

$$h_{i,i+1} = X_{i,i+1} + G_{i,i+1} + \bar{X}_{i,i+1} = G_{i,i+1},$$

using the vanishing of $X + \bar{X}$ on the superdiagonal. Since $h = I + (h - I)$ and $g = I + G$, this is $h_{i,i+1} = g_{i,i+1}$. $\square$

Lemma 3. *For $g \in U$, the Jordan block partition $\lambda(g)$ is well defined and is invariant under $GL_n(\mathbf{F}_p)$ -conjugation. Hence it is invariant under U -conjugation and D -conjugation.*

Proof. Since g is upper triangular with all diagonal entries equal to 1, $TI - g$ is upper triangular with diagonal entries $T - 1$, so the characteristic polynomial of g is $(T - 1)^n$. Let $\bar{\mathbf{F}}_p$ be an algebraic closure of $\mathbf{F}_p$. By the Jordan canonical form theorem over the algebraically closed field $\bar{\mathbf{F}}_p$, g is conjugate by $GL_n(\bar{\mathbf{F}}_p)$ to a unique (up to ordering of blocks) Jordan matrix $\text{diag}(J_{m_1}(1), \dots, J_{m_r}(1))$ with $m_1 + \dots + m_r = n$, where $J_m(1)$ is the $m \times m$ Jordan block with eigenvalue 1. Define $\lambda(g)$ to be the partition $(m_1, \dots, m_r)$ sorted in nonincreasing order. If $g_2 = zg_1z^{-1}$ with $z \in GL_n(\mathbf{F}_p)$, then the same equality holds in $GL_n(\bar{\mathbf{F}}_p)$, so g_1 and

g_2 have the same Jordan form over $\bar{\mathbf{F}}_p$ and hence the same partition. Since $U \subset GL_n(\mathbf{F}_p)$ and $D \subset GL_n(\mathbf{F}_p)$, the stated invariances follow. The same block sizes describe $g - I$, which is nilpotent with eigenvalue 0, since $J_m(1) - I = J_m(0)$. □

Lemma 4. *D normalizes U . For $d = \text{diag}(d_1, \dots, d_n) \in D$ and any matrix g ,*

$$(dgd^{-1})_{ij} = d_i d_j^{-1} g_{ij} \quad \text{for all } i, j.$$

In particular, if $g \in U$ then $dgd^{-1} \in U$, and $(dgd^{-1})_{i,i+1} = d_i d_{i+1}^{-1} g_{i,i+1}$.

Proof. Left multiplication by d multiplies row i by d_i , so $(dg)_{ij} = d_i g_{ij}$. Right multiplication by d^{-1} multiplies column j by d_j^{-1} , so $(dgd^{-1})_{ij} = d_i g_{ij} d_j^{-1}$. If $g \in U$, then $g_{ij} = 0$ for $i > j$ implies $(dgd^{-1})_{ij} = 0$ for $i > j$, and $g_{ii} = 1$ implies $(dgd^{-1})_{ii} = d_i d_i^{-1} \cdot 1 = 1$. Hence $dgd^{-1} \in U$. The superdiagonal formula is the special case $j = i + 1$. □

Lemma 5. *Fix $d \in D$. The map $\phi_d(g) = dgd^{-1}$ is an automorphism of U , and it permutes the U -conjugacy classes of U . If C is the U -class of g , then $\phi_d(C)$ is the U -class of $\phi_d(g)$. Moreover $\lambda(\phi_d(g)) = \lambda(g)$, $S(\phi_d(g)) = S(g)$, and $v(\phi_d(g)) = d \cdot v(g)$.*

Proof. By Lemma 4, ϕ_d maps U to U , with inverse $\phi_{d^{-1}}$, and it is a homomorphism since conjugation is. Hence it is an automorphism of U . Let $C = \{ugu^{-1} : u \in U\}$. Then

$$\phi_d(C) = \{dugu^{-1}d^{-1} : u \in U\} = \{(dud^{-1})(dgd^{-1})(dud^{-1})^{-1} : u \in U\}.$$

By Lemma 4, as u ranges over U , dud^{-1} ranges over all of U , because ϕ_d is surjective on U . Hence $\phi_d(C)$ equals the U -class of dgd^{-1} . The equality of Jordan partitions holds by Lemma 3 since $d \in GL_n(\mathbf{F}_p)$. By Lemma 4, $(dgd^{-1})_{i,i+1} = d_i d_{i+1}^{-1} g_{i,i+1}$ with $d_i d_{i+1}^{-1} \neq 0$, so the left side is zero iff $g_{i,i+1}$ is zero. Hence S is preserved and v transforms by $d \cdot$ as defined. □

Lemma 6. *The action $(d, v) \mapsto d \cdot v$ of D on $\mathbf{F}_p^{n-1}$ preserves each V_S and acts transitively on each V_S .*

Proof. For $v \in V_S$, $(d \cdot v)_i = d_i d_{i+1}^{-1} v_i$ with nonzero scalar $d_i d_{i+1}^{-1}$, so $(d \cdot v)_i \neq 0$ iff $v_i \neq 0$. Hence $d \cdot v \in V_S$. Let $v, w \in V_S$. For $i \in S$, $v_i, w_i \in \mathbf{F}_p^\times$, so $r_i = w_i v_i^{-1} \in \mathbf{F}_p^\times$ is defined. For $i \notin S$, set $r_i = 1$. Define $d_n = 1$ and $d_i = \prod_{j=i}^{n-1} r_j$ for $i < n$, with the empty product equal to 1 when $n = 1$. Each $d_i \in \mathbf{F}_p^\times$, so $d = \text{diag}(d_1, \dots, d_n) \in D$. By construction $d_i d_{i+1}^{-1} = r_i$ for all i . If $i \in S$, $(d \cdot v)_i = r_i v_i = w_i$. If $i \notin S$, $v_i = w_i = 0$, so $(d \cdot v)_i = 0 = w_i$. Hence $d \cdot v = w$. For $n = 1$, $V_\emptyset$ is a singleton and transitivity holds vacuously. □

Lemma 7. *For every S , V_S is nonempty and $|V_S| = (p-1)^{|S|}$.*

Proof. A vector $v \in \mathbf{F}_p^{n-1}$ belongs to V_S iff $v_i \in \mathbf{F}_p^\times$ for $i \in S$ and $v_i = 0$ for $i \notin S$. There are $p-1$ choices for each coordinate in S and one choice for each coordinate outside S . Hence $|V_S| = (p-1)^{|S|} \cdot 1^{n-|S|} = (p-1)^{|S|}$. The vector with 1 on S and 0 elsewhere belongs to V_S since $0 \neq 1$ in $\mathbf{F}_p$, so V_S is nonempty. □

Lemma 8. *Fix a partition λ of n and $S \subset \{1, \dots, n-1\}$. For $v \in \mathbf{F}_p^{n-1}$, let N_v be the number of U -conjugacy classes C with $\lambda(C) = \lambda$ and $v(C) = v$. If $v, w \in V_S$ then $N_v = N_w$.*

Proof. By Lemma 2, $v(g)$ is constant on U -classes, so $v(C)$ is well defined; by Lemma 3, $\lambda(C)$ is well defined. Hence N_v is well defined (finite since U is finite). Let $v, w \in V_S$. By Lemma 6 there exists $d \in D$ with $d \cdot v = w$. By Lemma 5, $C \mapsto \phi_d(C) = dCd^{-1}$ is a bijection of the set of all U -classes onto itself, with inverse $\phi_{d^{-1}}$. If $v(C) = v$ then $v(\phi_d(C)) = d \cdot v = w$, and $\lambda(\phi_d(C)) = \lambda(C)$. Hence ϕ_d restricts to a bijection between classes with invariants (λ, v) and classes with invariants (λ, w) . Therefore the two finite counts are equal. $\square$

Proof of the theorem. Fix n, p, λ, S . By Lemma 2 and Lemma 3, the properties $\lambda(g) = \lambda$ and $S(g) = S$ are constant on U -conjugacy classes, so $N_{n,p}(\lambda, S)$ counts classes C with $\lambda(C) = \lambda$ and $v(C) \in V_S$. With N_v as in Lemma 8,

$$N_{n,p}(\lambda, S) = \sum_{v \in V_S} N_v,$$

since the sets of classes with distinct values $v(C)$ are disjoint and their union over $v \in V_S$ is exactly the set counted by $N_{n,p}(\lambda, S)$.

By Lemma 7, V_S is nonempty; choose $v_0 \in V_S$ and let $m = N_{v_0}$. By Lemma 8, $N_v = m$ for every $v \in V_S$. Hence

$$N_{n,p}(\lambda, S) = |V_S| \cdot m = (p-1)^{|S|} \cdot m,$$

using Lemma 7 for $|V_S|$. Here m is a nonnegative integer ($m = 0$ is allowed, in which case the sum is 0). Therefore $(p-1)^{|S|}$ divides $N_{n,p}(\lambda, S)$ in the integers. For $n = 1$, $S = \emptyset$ and $(p-1)^0 = 1$, which divides every integer. This holds for every $n \geq 1$, prime p , partition λ of n , and $S \subset \{1, \dots, n-1\}$. $\square$

4. Computational verification

For Theorem 1, all matrices of $UT_n(\mathbf{F}_p)$ were enumerated for (n, p) in $\{(2, 2), (2, 3), (2, 5), (3, 2), (3, 3), (3, 5), (4, 2), (4, 3), (5, 2)\}$. $UT_n(\mathbf{F}_p)$ -conjugacy classes were computed by brute-force conjugation, and for each class representative the support S , Jordan type λ via nullities of powers of $A - I$, and centralizer order were tabulated. The count $N(n, p, S, \lambda, e)$ was checked for divisibility by $(p-1)^{|S|}$ for all S, λ, e . No counterexample was found. The run tested 102 total N -keys and reported sanity checks for rank versus brute-force nullspace, $UT_3(\mathbf{F}_2)$ order 8 and 5 classes, and $UT_2(\mathbf{F}_3)$ abelian with 3 classes as passed.

For Theorem 2, $UT_n(\mathbf{F}_p)$ was enumerated for $n \in \{2, 3, 4\}$ and $p \in \{2, 3\}$ and additionally $n = 3, p = 5$, over every partition λ and every subset S , computing Jordan type via ranks of powers of $g - I$ and support via the first superdiagonal, counting $N_{n,p}(\lambda, S)$, and verifying divisibility by $(p-1)^{|S|}$. No counterexample was found. The run tested 124 total cases over pairs $(2, 2), (2, 3), (3, 2), (3, 3), (4, 2), (4, 3), (3, 5)$ and reported sanity checks on order and partition, rank cross-check and identity Jordan form, and known class numbers as passed.

An independent check re-examined all lemmas and the theorem for each result and reported all checks passed.

Lean 4 status: Theorem 1 compiles with `sorry`. It has 3 sorries and depends on `sorryAx`, so it is not machine-checked sorry-free. Its back-translation was judged faithful: quantifiers, hypotheses and conclusion match, including $n \geq 2$, p prime, S contained in 1 to $n-1$, partition of n , support via 1-indexed superdiagonal, centralizer cardinality, and class count including $N = 0$ when empty, with no added hypotheses or trivialization. Theorem 2 failed to compile with a parse error at line 1, so it has no sorry-free proof and no back-translation faithfulness verdict.

5. Limitations and open questions

Computational ranges are small, with $n \leq 5$ and $p \leq 5$, so enumeration does not test large n , large p , or general q . The two theorems overlap, with Theorem 1 adding centralizer order and Theorem 2 allowing $n = 1$, and no commuting-pair, commuting-triple, branching, or commutator-membership identities from

the seed topic are proved here. The referee noted one minor typographical issue in the proof of Theorem 1, where the inverse class map was written as $E \mapsto D^{-1}ED^{-1}$ instead of $E \mapsto D^{-1}ED$, while the surrounding text correctly states that D^{-1} replaces D ; this is carried forward here and does not affect the argument. There are no remaining `## GAP` markers to carry forward. A sorry-free Lean proof would certify only the Lean statement as written, which must still be read against the informal statement, and no such proof is claimed here.

Novelty

Both results were judged apparently new on the basis of a bounded automated search over arXiv, Crossref, and OpenAlex indexes only. This is not a literature review, and it is blind to books, journals outside those indexes, preprints phrased differently, and any result not containing the searched terms. No evidence base is memory-only; both searches returned results.

For Theorem 1, the exact queries run were divisibility of the number of conjugacy classes of the finite unitriangular group $UT_n(F_q)$ with fixed superdiagonal support, Jordan block partition, and centralizer order by powers of $q - 1$, enumeration of conjugacy classes of unitriangular groups UT_n over finite fields, Higman conjecture on polynomial dependence on q , Jordan canonical form and centralizer sizes in unipotent groups, and diagonal torus conjugation action on UT_n scaling matrix entries and superdiagonal entries to count orbits of conjugacy classes in finite p -groups. They returned 27 hits in total, none matching, with closest known results being Higman-type polynomiality, torus normalization and superdiagonal invariance facts, and general divisibility theorems for class numbers. No search errors were reported.

For Theorem 2, the exact queries run were conjugacy classes of the finite unitriangular group $UT_n(F_q)$ with prescribed Jordan block partition and prescribed set of nonzero first superdiagonal entries, and divisibility of their number by powers of $q - 1$, enumeration and classification of conjugacy classes of maximal unipotent subgroups of GL_n over finite fields, algebra groups and pattern subgroups, including counts refined by Jordan canonical form and Higman and Lehrer conjectures, and conjugation action of the diagonal torus normalizing the unitriangular group on its conjugacy classes and coadjoint orbits by scaling matrix entries, used to count orbits with given support. They returned 30 hits in total, none matching, with closest known results being torus normalization, superdiagonal invariance, general UT_n enumeration and Higman and Lehrer conjectures, and torus actions on coadjoint orbits and superclasses. No search errors were reported.

Open questions include whether similar torus divisibility holds for finer invariants such as full superdiagonal value patterns beyond support, higher diagonals, commutator membership, or commuting-tuple counts, and whether the common fiber multiplicity m admits a closed formula.