First-Entry Deletion Recurrence for Up-Down Permutations Refined by Last Entry and Record Positions

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Text generated by language models in the mathforge pipeline; not reviewed by a person.

Abstract

Up-down counts with fixed first entry, last entry, and record-position set equal summed down-up counts one size smaller. For $n \geq 2$, $a, b \in [n]$ with $b \neq a$, and $S \subset [n]$ with $1, 2 \in S$, let $b^\star$ be the standardized last value and $S^\star$ the shifted record set. Then

$$N_{ud}(n, a, b, S) = \sum_{c=a}^{n-1} N_{du}(n-1, c, b^\star, S^\star).$$

The proof is an elementary bookkeeping refinement of the classical first-entry deletion and standardization bijection. Reported exhaustive checks cover $2 \leq n \leq 8$; the attempted Lean formalization did not compile.

1. Introduction

Context and scope

Alternating up-down and down-up permutations are classically enumerated by Euler and Entringer numbers. First-entry deletion followed by order-isomorphism standardization underlies the classical Entringer recurrence. This paper tracks, in addition to the first entry, the last entry and the exact set of left-to-right record positions.

The contribution is modest and specialized: it is a clean, self-contained bookkeeping refinement of a standard bijection, not a new deletion method. The main point is that the initial ascent makes the deleted value irrelevant to every subsequent prefix maximum. Consequently, record positions shift without additional conditions on later entries.

Positioning and novelty

The joint recurrence is, within the bounded search described in Section 5, apparently new. This is not an established priority claim. No broader literature search has been performed for this revision.

The review suggested comparison with Entringer/Seidel treatments and sources including Stanley's EC1/2, Foata and Schützenberger, Foata and Han, Dumont, and Bona's *Combinatorics of Permutations*. These works were not consulted; no source-specific theorem, indexing convention, or comparison is asserted here. Our indexing is by permutation size n and actual first value a, with the two alternating orientations distinguished explicitly.

2. Notation

Write $[n] = \{1, \dots, n\}$, and let $\mathfrak{S}_n$ denote its permutations. Define

$$UD_n = \{\pi \in \mathfrak{S}_n : \pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots\},$$

and define DU_n by reversing all these inequalities. Set $UD_1 = DU_1 = \{[1]\}$.

The record-position set is

$$\mathrm{Rec}(\pi) = \{i \in [n] : \pi_i = \max_{j \le i} \pi_j\}.$$

Thus position 1 is always a record. Define

$$N_{ud}(n, a, b, S) = \big|\{\pi \in UD_n : \pi_1 = a,\ \pi_n = b,\ \mathrm{Rec}(\pi) = S\}\big|,$$

and define N_{du} similarly.

For fixed $n \ge 2$ and $a \in [n]$, the notation used throughout is:

$$\phi(x) = \begin{cases} x, & x < a, \\ x - 1, & x > a, \end{cases} \qquad \psi(y) = \begin{cases} y, & y < a, \\ y + 1, & y \ge a. \end{cases}$$

Here ϕ removes the missing value a, and ψ restores it. The maps F and G delete and prepend the first entry, respectively, while applying these value maps. We write $b^\star = \phi(b)$, $S^\star$ for the shifted record set, and $c = F(\pi)_1$ for the standardized second value.

3. Results

Theorem 1 (First-entry deletion recurrence for up-down permutations refined by last entry and record positions)**.** *For every integer $n \ge 2$, every $a \in [n]$, every $b \in [n]$ with $b \ne a$, and every $S \subset [n]$ with $1 \in S$ and $2 \in S$, let*

$$b^\star = \begin{cases} b - 1, & b > a, \\ b, & b < a, \end{cases} \qquad S^\star = \{1\} \cup \{i : 2 \le i \le n - 1,\ i + 1 \in S\}.$$

Then

$$N_{ud}(n, a, b, S) = \sum_{c=a}^{n-1} N_{du}(n - 1, c, b^\star, S^\star),$$

where the empty sum is 0*.*

Fix $n \ge 2$ and $a \in [n]$ for the following lemmas.

Lemma 1 (Standardization of values)**.** *The maps*

$$\phi : [n] \setminus \{a\} \longrightarrow [n - 1], \qquad \psi : [n - 1] \longrightarrow [n] \setminus \{a\}$$

defined in Section 2 are mutually inverse bijections. For $x, y \ne a$,

$$x < y \quad \Longleftrightarrow \quad \phi(x) < \phi(y).$$

Proof. If $x < a$, then $\psi(\phi(x)) = x$. If $x > a$, then $x - 1 \ge a$, so $\psi(\phi(x)) = (x - 1) + 1 = x$. Conversely, if $y < a$, then $\phi(\psi(y)) = y$; if $y \ge a$, then $\phi(\psi(y)) = (y + 1) - 1 = y$.

For $x < y$, order preservation is immediate when both lie below a or both lie above it. In the remaining case $x < a < y$,

$$\phi(x) = x \le a - 1 < a \le y - 1 = \phi(y).$$

The converse follows from strict order preservation and injectivity.

□

In particular, ϕ maps $\{a + 1, \dots, n\}$ onto $\{a, \dots, n - 1\}$.

Lemma 2 (Deletion bijection)**.** *Define*

$$F : \{\pi \in \mathfrak{S}_n : \pi_1 = a\} \longrightarrow \mathfrak{S}_{n-1}, \qquad F(\pi)_j = \phi(\pi_{j+1}),$$

and

$$G : \mathfrak{S}_{n-1} \longrightarrow \{\pi \in \mathfrak{S}_n : \pi_1 = a\}, \qquad G(\sigma)_1 = a, \quad G(\sigma)_{j+1} = \psi(\sigma_j).$$

Then F and G are mutually inverse bijections.

Proof. The tail of π lists each value of $[n] \setminus \{a\}$ exactly once. Lemma 1 therefore makes $F(\pi)$ a permutation of $[n-1]$. Conversely, applying ψ to a permutation of $[n-1]$ and prepending a lists every value of $[n]$ exactly once. The inverse identities hold entrywise by Lemma 1.

□

Lemma 3 (Pattern and first value)**.** *Let π, with $\pi_1 = a$, and $\sigma = F(\pi)$ correspond as in Lemma 2. For $1 \le j \le n-2$,*

$$\sigma_j < \sigma_{j+1} \quad \Longleftrightarrow \quad \pi_{j+1} < \pi_{j+2}.$$

Consequently, the DU_{n-1} inequalities for σ are equivalent to all the UD_n inequalities for π except possibly the first. Moreover,

$$\pi_1 < \pi_2 \quad \Longleftrightarrow \quad \sigma_1 \ge a.$$

Thus

$$\pi \in UD_n \quad \Longleftrightarrow \quad \sigma \in DU_{n-1} \text{ and } \sigma_1 \ge a.$$

Proof. The comparison equivalence follows from Lemma 1. Shifting the position index by one reverses its parity, so a UD tail becomes DU. When $n = 2$, both tail conditions are vacuous.

Finally, $\pi_2 = \psi(\sigma_1)$. If $\sigma_1 \ge a$, this equals $\sigma_1 + 1 > a$; otherwise it equals $\sigma_1 < a$. This proves the first-inequality equivalence and the membership assertion.

□

The lower bound is inclusive: $c = a$ corresponds to $\pi_2 = a+1$, not to an equality of adjacent original entries.

Lemma 4 (Last value)**.** *Assume $b \ne a$, and set $b^\star = \phi(b)$. Then $b^\star \in [n-1]$, and for corresponding π, σ,*

$$\pi_n = b \quad \Longleftrightarrow \quad \sigma_{n-1} = b^\star.$$

Proof. If $b > a$, then $1 \le b-1 \le n-1$; if $b < a$, then $1 \le b \le n-1$. Since $\sigma_{n-1} = \phi(\pi_n)$, the equivalence follows from injectivity of ϕ.

□

Lemma 5 (Records)**.** *Let $\pi_1 = a$, assume $\pi_2 > \pi_1$, and let $\sigma = F(\pi)$. For every $j \in [n-1]$,*

$$j \in \mathrm{Rec}(\sigma) \quad \Longleftrightarrow \quad j+1 \in \mathrm{Rec}(\pi).$$

Hence

$$\mathrm{Rec}(\sigma) = \{j \in [n-1] : j+1 \in \mathrm{Rec}(\pi)\}.$$

Proof. Order preservation implies that σ_j is maximal among $\sigma_1, \dots, \sigma_j$ exactly when π_{j+1} is maximal among $\pi_2, \dots, \pi_{j+1}$. Since $j \ge 1$, this latter prefix contains $\pi_2 > a = \pi_1$. Adding π_1 therefore does not change its maximum. The two record conditions are equivalent.

□

Lemma 6 (Shifted record set)**.** *Let $S \subset [n]$ with $1, 2 \in S$, and define*

$$S^\star = \{1\} \cup \{i : 2 \le i \le n-1,\ i+1 \in S\}.$$

Then $S^\star \subset [n-1]$ and

$$S^\star = \{j \in [n-1] : j+1 \in S\}.$$

For π, σ as in Lemma 5,

$$\mathrm{Rec}(\pi) = S \quad \Longleftrightarrow \quad \mathrm{Rec}(\sigma) = S^\star.$$

Proof. The containment follows from the defining bounds. The shifted set contains 1 because $2 \in S$, and its membership condition at every $j \ge 2$ is precisely $j+1 \in S$. This proves the set identity.

The forward implication follows from Lemma 5. Conversely, that lemma recovers all record positions $2, \dots, n$ from $S^\star$. Position 1 is always a record and belongs to S, so the full original record set is S.

□

Proof of Theorem 1. Let

$$A = \{\pi \in UD_n : \pi_1 = a,\ \pi_n = b,\ \mathrm{Rec}(\pi) = S\},$$

and, for $c \in [n-1]$, let

$$B_c = \{\sigma \in DU_{n-1} : \sigma_1 = c,\ \sigma_{n-1} = b^\star,\ \mathrm{Rec}(\sigma) = S^\star\}.$$

The B_c are pairwise disjoint.

If $a = n$, the required first ascent is impossible, so A is empty. The stated sum is also empty. Suppose now that $a < n$.

1. **Forward map.** If $\pi \in A$, Lemma 3 gives $F(\pi) \in DU_{n-1}$ with first value $c \in \{a, \dots, n-1\}$. Lemma 4 gives its last value $b^\star$. The initial ascent permits application of Lemmas 5 and 6, giving its record set $S^\star$. Thus $F(\pi) \in B_c$.
2. **Inverse map.** If $\sigma \in B_c$ with $a \le c \le n-1$, Lemma 3 gives $G(\sigma) \in UD_n$, including its first ascent. Lemma 4 restores last value b, and Lemma 6 restores record set S. Thus $G(\sigma) \in A$.
3. **Exhaustion and counting.** Lemma 2 makes these restricted maps inverse. Therefore

$$|A| = \left| \bigsqcup_{c=a}^{n-1} B_c \right| = \sum_{c=a}^{n-1} |B_c|,$$

which is the asserted identity.

□

Assumptions and small sizes

Lemmas 1–3 require only $n \ge 2$, a fixed a, and the specified permutation correspondence. Lemma 4 additionally requires $b \ne a$ to define $\phi(b)$. Lemma 5 needs the initial ascent but not the remaining alternating inequalities. Lemma 6 needs $1, 2 \in S$ and the hypotheses of Lemma 5.

For actual permutations of size at least two, first and last values differ automatically. Nevertheless, the theorem specifies external counting parameters, and $b = a$ would make $b^\star = \phi(b)$ undefined. Similarly, $1, 2 \in S$ are consistency conditions for a UD_n record set. Without $2 \in S$, the left count is zero, but the displayed definition of $S^\star$ still inserts 1, so the same formula need not hold.

At $n = 2$, the sole up-down permutation is $(1, 2)$; deletion gives (1), with $b^\star = 1$ and $S^\star = \{1\}$. At $n = 3$, the same argument has just one tail comparison. No distinct entries, comparisons, or record positions are omitted in either case.

Worked example and map pseudocode

Take $n = 5$, $a = 2$, and

$$\pi = (2, 5, 1, 4, 3) \in UD_5.$$

Its last value is $b = 3$, and $\mathrm{Rec}(\pi) = \{1, 2\}$. Deleting the first entry and standardizing gives

$$(2, 5, 1, 4, 3) \longmapsto (5, 1, 4, 3) \longmapsto (4, 1, 3, 2) = \sigma.$$

Here $\sigma \in DU_4$, $c = 4 \geq 2$, $b^\star = 2$, and $\mathrm{Rec}(\sigma) = S^\star = \{1\}$. Applying ψ restores $(5, 1, 4, 3)$, and prepending 2 restores π.

The maps can be implemented directly:

$$\begin{aligned} F(\pi, a): &\quad \text{require } \pi \in \mathfrak{S}_n,\ \pi_1 = a;\ \text{return } [\phi(\pi_2), \dots, \phi(\pi_n)], \\ G(\sigma, a): &\quad \text{require } \sigma \in \mathfrak{S}_{n-1};\ \text{return } [a, \psi(\sigma_1), \dots, \psi(\sigma_{n-1})]. \end{aligned}$$

Relation to first-entry counting

The baseline relationship requires no new enumerative theorem. For fixed a, the map $b \mapsto b^\star$ bijects $[n] \setminus \{a\}$ with $[n-1]$. Likewise, $S \mapsto S^\star$ bijects subsets of $[n]$ containing $1, 2$ with subsets of $[n-1]$ containing 1; its inverse is

$$T \longmapsto \{1\} \cup \{j + 1 : j \in T\}.$$

Thus summing the displayed identity over its admissible b, S simply removes the two extra refinements:

$$\begin{aligned} |\{\pi \in UD_n : \pi_1 = a\}| &= \sum_{\substack{b \in [n] \setminus \{a\} \\ S \subset [n],\, 1, 2 \in S}} N_{ud}(n, a, b, S) \\ &= \sum_{c=a}^{n-1} \sum_{\substack{d \in [n-1] \\ T \subset [n-1],\, 1 \in T}} N_{du}(n-1, c, d, T) \\ &= \sum_{c=a}^{n-1} |\{\sigma \in DU_{n-1} : \sigma_1 = c\}|. \end{aligned}$$

This is the first-entry deletion relation in our size-and-value indexing. No source-specific Entringer or Seidel array convention is claimed.

4. Computational verification

The reported search space was exhaustive enumeration. For each n with $2 \leq n \leq 8$, all $n!$ permutations of $[n]$ were enumerated. Each was classified as UD_n or DU_n; first value, last value, and record set were computed; and the corresponding counts were tabulated. The transformed parameters and both sides of the recurrence were then evaluated for admissible parameter tuples.

The falsification run exited with code 0. Sanity checks confirmed the Euler totals

$$UD = DU = \{1 : 1,\ 2 : 1,\ 3 : 2,\ 4 : 5,\ 5 : 16,\ 6 : 61,\ 7 : 272,\ 8 : 1385\},$$

along with hand values and record and classifier cross-checks. No counterexample was found for $n = 2, \dots, 8$, with **5630 cases tested**.

The number 5630 is the reported case count, not a count of enumerated permutations. The retained description does not explain its precise accounting or how zero-count parameter tuples were represented; it should not be read as a newly audited count of all admissible tuples.

The independent check exited with code 0 and confirmed Lemmas 1–6 and the theorem sum identity, with all checks passed.

A schematic reproduction procedure is:

1. Enumerate permutations at each required size, including the size-one base case.
2. Tabulate counts keyed by orientation, first value, last value, and record set.
3. For each admissible (n, a, b, S), compare the UD count with the stated sum of DU counts.

This describes the mathematical test, not the exact original test harness or its case-count bookkeeping. No repository, raw-data archive, or reproduction commands are supplied here.

Lean 4 status for Theorem 1: failed to compile. The formalization exited with code 1, with error `unexpected identifier; expected command`, after three repairs. Its recorded status is `compiles: false` and `sorry_free: false`. There is no machine-checked sorry-free proof. Because no Lean statement compiled, there is no back-translation faithfulness verdict to report. The failed attempt is not evidence of correctness.

No computation, formalization, or literature search has been run since the paper was written.

5. Limitations and open questions

The computational evidence is bounded to $n \leq 8$; it is not a proof for larger sizes. The general argument is the bijective proof in Section 3, not an extrapolation from those tests. The review identified no specific proof error. Independent checking of the exposition and successful mechanization remain desirable.

The original text was generated by language models within an automated pipeline. The reported computational checks do not establish that a person independently checked every proof step. This revision does not claim such checking.

Open tasks raised by the review

- **Literature coverage.** A focused manual review, precise citations, source-specific indexing comparisons, and assessment against other Entringer/Seidel refinements remain open. The suggested works were not consulted.
- **Larger tests and artifacts.** Exhaustive tests at $n = 9, 10, 11$, randomized larger-size checks, public scripts and raw outputs, a permanent archive, reproduction commands, and continuous integration were requested by the review. None has been produced in this revision. The retained record does not document why 8 was chosen or any attempts beyond it.
- **Case-count audit.** Recovering the original harness and explaining the exact 5630-case accounting remain open.
- **Formalization.** A compiling, sorry-free proof remains open. The available status does not identify successfully encoded individual lemmas or diagnose the syntax error beyond its recorded message. Posting sources and logs, splitting the proof into smaller lemmas, and documenting the environment were suggested by the review but have not been completed.
- **Further consequences.** Fixed-last-value and fixed-record-set marginal recurrences, record-count distributions, generating functions, closed forms, numerical refinement tables, and a nontrivial enumerative application were requested by the review. They are not established here. The aggregation in Section 3 only explains the existing theorem's baseline relationship.
- **Other statistics and symmetries.** Extensions to minima, descent sets, reversal, complement, or other bijections remain unproved. Complement and reversal equinumerosity, parity and modular congruences, and other refinements in the seed topic are not proved here.
- **Algorithmic development.** The theorem specifies a summation once smaller down-up counts are available. A complete recursive counting algorithm, complexity analysis, and implementation remain open.

The identity's elementary nature limits its novelty claim. The review's optional strengthening projects would require additional results or verification, not merely changes in exposition.

Novelty

The automated novelty verdict was **APPARENTLY_NEW**, with closest known results listed as Entringer recurrence for alternating permutations by deletion of the first entry and order-isomorphism standardization, and distribution of left-to-right maxima and joint maxima and minima statistics on up-down and down-up permutations.

Exact queries run were:

Queries (3): 1. `recurrence number of up-down alternating permutations with given first letter last letter and left-to-right maxima set in terms of down-up permutations of length n-1` 2. `joint distribution of first value last value and records left-to-right maxima in alternating permutations Entringer numbers refinement` 3. `enumeration of alternating permutations by deletion of first entry and order-isomorphism standardization recurrence`

Search terms (3): 1. `Entringer numbers first last entry` 2. `alternating permutations left-to-right maxima record set recurrence` 3. `up-down permutations standardization deletion first element`

Follow-up queries (2): 1. `Entringer numbers joint distribution first entry last entry left-to-right maxima` 2. `alternating permutations deletion of first entry standardization records recurrence`

Retrieval returned 27 hits in total from arXiv and Crossref, and OpenAlex searches failed with HTTP 503 Service Unavailable for all three queries, so OpenAlex contributed 0 hits. Evidence base is retrieval, not memory-only. This is a bounded automated search over those indexes, not a literature review, and is blind to books, journals outside them, and anything phrased differently.

6. Changes in response to review

- **Strength 1—joint refinement:** Retained the theorem and bounded novelty verdict while distinguishing the bookkeeping contribution from the classical bijection.
- **Strength 2—elementary methodology:** Retained all six lemmas and organized the theorem proof into forward map, inverse map, and counting.
- **Strength 3—technical details:** Clarified the assumptions, inclusive lower bound, empty sum, and small-size cases.
- **Strength 4—verification:** Preserved the reported ranges and statuses, while separating computational evidence from proof and qualifying the unexplained case count.
- **Strength 5—modularity:** Consolidated notation and made the maps' domains and inverse relationship explicit.
- **Weakness 1 and suggestion 1—positioning:** Tempered the introduction and explained the baseline aggregation; broader comparison remains open, and suggested sources are explicitly unconsulted.
- **Weakness 2 and suggestion 2—verification:** Expanded the general proof's coverage and stated that failed Lean compilation is not evidence. Larger tests, artifacts, and successful formalization remain open.
- **Weakness 3 and suggestion 3—consequences:** Added the direct first-entry aggregation with explicit indexing. New marginals, tables, generating functions, and nontrivial applications remain open.
- **Weakness 4 and suggestion 4—presentation:** Standardized notation, removed typographical artifacts, added a worked example and map pseudocode, and clarified provenance. No unavailable repository or scripts are claimed.
- **Suggestion 5—stronger novelty:** Adopted the modest bookkeeping-refinement description; stronger enumerative results and full formalization remain open.
- **Checklist 1—literature comparison:** Added scope and indexing discussion; precise literature comparison could not be supplied without consultation.

- **Checklist 2—reductions:** Explained first-entry aggregation; an additional marginal result remains open.
- **Checklist 3—example:** Added a size-five example tracing both maps, last value, and records.
- **Checklist 4—public artifacts:** Recorded publication and reproduction infrastructure as open rather than inventing links.
- **Checklist 5—Lean:** Retained the required failed-compilation and faithfulness statuses candidly; completion and CI remain open.
- **Checklist 6—copy-editing:** Reworked formatting, terminology, and cross-references.
- **Checklist 7—utility or modesty:** Tempered novelty instead of claiming an unestablished application.
- **Question 1—assumptions:** Explained their role lemma by lemma, including why $\phi(b)$ requires $b \neq a$ and why omitting $2 \in S$ can invalidate the displayed formula.
- **Question 2—equality case:** Used $c = a, \ldots, n-1$ consistently and explained that $c = a$ restores second value $a + 1$.
- **Question 3—5630 cases and cutoff:** Distinguished the reported case count from permutation enumeration, supplied schematic enumeration steps, and identified unavailable accounting, cutoff rationale, and artifacts.
- **Question 4—formalization details:** Preserved the exact recorded error and status; unavailable lemma-level information and source publication remain open.
- **Question 5—extensions:** Listed the proposed statistics and enumerative applications as unproved directions rather than new results.
- **Dimension scores and overall assessment:** Reflected the review's assessment of elementary, specialized scope without altering the theorem or verification claims.

Models

First-entry deletion recurrence for up-down permutations refined by last entry and record positions (`verified`)

- Proposed the claim: muse-spark-1.3-contributor-free
- Counterexample search: muse-spark-1.3-contributor-free
- Wrote the proof: muse-spark-1.3-contributor-free
- Referee: muse-spark-1.3-contributor-free
- Independent check script: muse-spark-1.3-contributor-free
- Lean formalization: muse-spark-1.3-contributor-free
- Faithfulness judge: muse-spark-1.3-contributor-free