

First-Entry Deletion Recurrence for Up-Down Permutations Refined by Last Entry and Record Positions

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Abstract

Up-down counts with fixed first entry and record set equal summed down-up counts one size smaller. For $n \geq 2$, $a \in [n]$, $b \in [n]$ with $b \neq a$, and $S \subset [n]$ with $1 \in S$ and $2 \in S$, with b^* the standardized last value and S^* the shifted record set, $N_{ud}(n, a, b, S) = \sum_{c=a}^{n-1} N_{du}(n-1, c, b^*, S^*)$.

1. Introduction

Context

Alternating up-down and down-up permutations of $\{1, \dots, n\}$ are classically enumerated by Euler and Entringer numbers. Entringer theory refines alternating permutations by first entry via deletion of the first entry and order-isomorphism standardization. Record statistics, namely left-to-right maxima, are studied separately.

What is new

This paper proves a joint refinement by first entry, last entry, and full record-position set, with explicit shifted set S^* , standardized last value b^* , and summation over standardized second value c . The deletion and standardization bijection itself is standard and underlies the classical Entringer recurrence. The stated joint recurrence for first value, last value, and full record set is, within the bounded search described in Section 5, apparently new.

2. Notation

$[n]$ is $\{1, \dots, n\}$. UD_n is the set of permutations π of $[n]$ with $\pi_1 < \pi_2 > \pi_3 < \pi_4$ and so on. DU_n is the set of permutations π of $[n]$ with $\pi_1 > \pi_2 < \pi_3 > \pi_4$ and so on, with $UD_1 = DU_1 = \{[1]\}$. $\text{Rec}(\pi)$ is $\{i \in [n] : \pi_i = \max_{j \leq i} \pi_j\}$. $N_{ud}(n, a, b, S)$ is the number of π in UD_n with $\pi_1 = a$ and $\pi_n = b$ and $\text{Rec}(\pi) = S$. $N_{du}(n, a, b, S)$ is the number of π in DU_n with $\pi_1 = a$ and $\pi_n = b$ and $\text{Rec}(\pi) = S$. $b^* \in [n-1]$ is the standardized last value. $S^* \subset [n-1]$ is the shifted record set. c is the standardized second value.

3. Results

Theorem 1 (First-entry deletion recurrence for up-down permutations refined by last entry and record positions). *For every integer $n \geq 2$, every $a \in [n]$, every $b \in [n]$ with $b \neq a$, and every $S \subset [n]$ with $1 \in S$ and $2 \in S$, let*

$$b^* = \begin{cases} b-1 & b > a \\ b & b < a \end{cases}, \quad S^* = \{1\} \cup \{i : 2 \leq i \leq n-1, i+1 \in S\},$$

then

$$N_{ud}(n, a, b, S) = \sum_{c=a}^{n-1} N_{du}(n-1, c, b^*, S^*),$$

where the empty sum is 0.

Here $[n] = \{1, \dots, n\}$, UD_n is the set of permutations π of $[n]$ with $\pi_k < \pi_{k+1}$ for k odd and $\pi_k > \pi_{k+1}$ for k even, $1 \leq k < n$, DU_n is the set with the reverse inequalities $\pi_k > \pi_{k+1}$ for k odd and $\pi_k < \pi_{k+1}$ for k even, $UD_1 = DU_1 = \{[1]\}$, $\text{Rec}(\pi) = \{i : \pi_i = \max_{j \leq i} \pi_j\}$, and N_{ud}, N_{du} count permutations with given first value, last value, and record set.

Fix $n \geq 2$ and $a \in [n]$ throughout.

Lemma 1 (standardization of values). Define $\phi : [n] \setminus \{a\} \rightarrow [n-1]$ by $\phi(x) = x$ if $x < a$ and $\phi(x) = x-1$ if $x > a$. Define $\psi : [n-1] \rightarrow [n] \setminus \{a\}$ by $\psi(y) = y$ if $y < a$ and $\psi(y) = y+1$ if $y \geq a$. Then ϕ and ψ are mutually inverse bijections, and for $x, y \neq a$, $x < y$ if and only if $\phi(x) < \phi(y)$.

Proof. If $x < a$ then $\psi(\phi(x)) = \psi(x) = x$ since $x < a$. If $x > a$ then $\phi(x) = x-1 \geq a$, so $\psi(\phi(x)) = (x-1)+1 = x$. Hence $\psi \circ \phi$ is the identity. If $y < a$ then $\psi(y) = y < a$, so $\phi(\psi(y)) = \phi(y) = y$. If $y \geq a$ then $\psi(y) = y+1 > a$, so $\phi(\psi(y)) = (y+1)-1 = y$. Hence $\phi \circ \psi$ is the identity. Therefore both are bijections and inverses of each other.

For monotonicity, let $x < y$, both different from a . If both are $< a$, $\phi(x) = x < y = \phi(y)$. If both are $> a$, $\phi(x) = x-1 < y-1 = \phi(y)$. If $x < a < y$, then $\phi(x) = x \leq a-1$ while $\phi(y) = y-1 \geq a$, so $\phi(x) < \phi(y)$. Thus $x < y$ implies $\phi(x) < \phi(y)$. Since ϕ is injective, $x > y$ implies $\phi(x) > \phi(y)$, and $x = y$ implies equality. Hence the equivalence holds. □

In particular ϕ maps $\{a+1, \dots, n\}$ bijectively onto $\{a, \dots, n-1\}$.

Lemma 2 (deletion bijection). Define $F : \{\pi \in S_n : \pi_1 = a\} \rightarrow S_{n-1}$ by $F(\pi)_j = \phi(\pi_{j+1})$ for $1 \leq j \leq n-1$, and $G : S_{n-1} \rightarrow \{\pi \in S_n : \pi_1 = a\}$ by $G(\sigma)_1 = a$ and $G(\sigma)_{j+1} = \psi(\sigma_j)$ for $1 \leq j \leq n-1$. Then F and G are mutually inverse bijections.

Proof. If $\pi_1 = a$ then $(\pi_2, \dots, \pi_n)$ is an ordering of $[n] \setminus \{a\}$, so applying ϕ entrywise yields an ordering of $[n-1]$ by Lemma 1, hence a permutation. Conversely, if σ is a permutation of $[n-1]$, applying ψ entrywise yields an ordering of $[n] \setminus \{a\}$, so prepending a yields a permutation of $[n]$ with first entry a . Since ϕ, ψ are inverses entrywise, F and G are inverses. □

Lemma 3 (pattern and first value). Let π with $\pi_1 = a$ and σ correspond by Lemma 2. Then for $1 \leq j \leq n-2$, $\sigma_j < \sigma_{j+1}$ if and only if $\pi_{j+1} < \pi_{j+2}$. Consequently σ satisfies the DU_{n-1} inequalities $\sigma_j > \sigma_{j+1}$ for j odd, $\sigma_j < \sigma_{j+1}$ for j even, if and only if π satisfies all UD_n inequalities except possibly the first, namely $\pi_{j+1} > \pi_{j+2}$ for j odd and $\pi_{j+1} < \pi_{j+2}$ for j even. Moreover $\pi_1 < \pi_2$ if and only if $\sigma_1 \geq a$. Therefore $\pi \in UD_n$ if and only if $\sigma \in DU_{n-1}$ and $\sigma_1 \geq a$.

Proof. By Lemma 1, order is preserved: $\sigma_j < \sigma_{j+1}$ means $\phi(\pi_{j+1}) < \phi(\pi_{j+2})$, which holds if and only if $\pi_{j+1} < \pi_{j+2}$. Since permutations have distinct entries, the reverse inequality also corresponds. The DU_{n-1} condition is that for $1 \leq j \leq n-2$, $\sigma_j > \sigma_{j+1}$ when j is odd and $\sigma_j < \sigma_{j+1}$ when j is even; equivalently $\sigma_j < \sigma_{j+1}$ if and only if j is even. The tail UD condition on π is that for $1 \leq j \leq n-2$, $\pi_{j+1} < \pi_{j+2}$ if and only if $j+1$ is odd, i.e. if and only if j is even. By the entrywise correspondence these two conditions are equivalent. When $n = 2$ there is no j and both conditions are vacuous.

For the first inequality, $\pi_2 = \psi(\sigma_1)$. If $\sigma_1 \geq a$ then $\psi(\sigma_1) = \sigma_1 + 1 > a = \pi_1$. If $\sigma_1 < a$ then $\psi(\sigma_1) = \sigma_1 < a = \pi_1$. Hence $\pi_1 < \pi_2$ holds if and only if $\sigma_1 \geq a$. Full UD_n membership is tail inequalities plus $\pi_1 < \pi_2$, hence equivalent to $\sigma \in DU_{n-1}$ plus $\sigma_1 \geq a$.

□

Lemma 4 (last value). Assume $b \neq a$ and define $b^* = \phi(b)$, i.e. $b - 1$ if $b > a$ and b if $b < a$. Then $b^* \in [n - 1]$. Let π, σ correspond as above. Then $\pi_n = b$ if and only if $\sigma_{n-1} = b^*$.

Proof. If $b > a$ then $2 \leq b \leq n$ and $b^* = b - 1 \in [1, n - 1]$. If $b < a$ then $1 \leq b \leq n - 1$ since $b < a \leq n$, so $b^* = b \in [1, n - 1]$. By definition $\sigma_{n-1} = \phi(\pi_n)$. Since ϕ is injective with $\phi(b) = b^*$, $\pi_n = b$ holds if and only if $\phi(\pi_n) = \phi(b)$, i.e. $\sigma_{n-1} = b^*$.

□

Lemma 5 (records). Let π with $\pi_1 = a$ and $\pi_2 > \pi_1$, and let $\sigma = F(\pi)$. Then for every $j \in [n - 1]$, $j \in \text{Rec}(\sigma)$ if and only if $j + 1 \in \text{Rec}(\pi)$. Hence $\text{Rec}(\sigma) = \{j \in [n - 1] : j + 1 \in \text{Rec}(\pi)\}$.

Proof. By definition $j \in \text{Rec}(\sigma)$ means $\sigma_j = \max_{k \leq j} \sigma_k$, and $j + 1 \in \text{Rec}(\pi)$ means $\pi_{j+1} = \max_{k \leq j+1} \pi_k$. By Lemma 1, for $k \leq j$, $\sigma_k \leq \sigma_j$ if and only if $\pi_{k+1} \leq \pi_{j+1}$, because $\sigma_k = \phi(\pi_{k+1})$. Thus σ_j is the maximum of $\sigma_1, \dots, \sigma_j$ if and only if π_{j+1} is the maximum of $\pi_2, \dots, \pi_{j+1}$.

Write $M' = \max_{k=2}^{j+1} \pi_k$ and $M = \max_{k=1}^{j+1} \pi_k = \max(\pi_1, M')$. Since $j \geq 1$, $M' \geq \pi_2 > \pi_1$ by hypothesis, so $M = M'$. Therefore $\pi_{j+1} = M'$ if and only if $\pi_{j+1} = M$, which is exactly the claimed equivalence.

□

Lemma 6 (shifted record set). Let $S \subset [n]$ with $1, 2 \in S$ and $S^* = \{1\} \cup \{i : 2 \leq i \leq n - 1, i + 1 \in S\}$. Then $S^* \subset [n - 1]$ and $S^* = \{j \in [n - 1] : j + 1 \in S\}$. Moreover, for π, σ as in Lemma 5, $\text{Rec}(\pi) = S$ if and only if $\text{Rec}(\sigma) = S^*$.

Proof. S^* is contained in $[n - 1]$ by construction. Let $T = \{j \in [n - 1] : j + 1 \in S\}$. Then $1 \in T$ because $2 \in S$. For $j \geq 2$, $j \in T$ if and only if $j + 1 \in S$, which is the defining condition of S^* for $j \geq 2$. Hence $T = S^*$.

If $\text{Rec}(\pi) = S$, Lemma 5 gives $\text{Rec}(\sigma) = \{j : j + 1 \in S\} = S^*$. Conversely, suppose $\text{Rec}(\sigma) = S^*$. By Lemma 5, for $j \geq 1$, $j + 1 \in \text{Rec}(\pi)$ if and only if $j \in S^*$. Hence $\text{Rec}(\pi) \cap \{2, \dots, n\} = \{j + 1 : j \in S^*\}$. Now $\{j + 1 : j \in S^*\} = \{2\} \cup (S \cap [3, n])$. Since $2 \in S$, this equals $S \cap [2, n]$. Also $1 \in \text{Rec}(\pi)$ always and $1 \in S$ by hypothesis. Hence $\text{Rec}(\pi) = S$.

□

Proof of the theorem. Let $A = \{\pi \in UD_n : \pi_1 = a, \pi_n = b, \text{Rec}(\pi) = S\}$ so $|A| = N_{ud}(n, a, b, S)$, and for $c \in [n - 1]$ let $B_c = \{\sigma \in DU_{n-1} : \sigma_1 = c, \sigma_{n-1} = b^*, \text{Rec}(\sigma) = S^*\}$ so $|B_c| = N_{du}(n - 1, c, b^*, S^*)$. The sets B_c are pairwise disjoint.

If $a = n$, no $\pi \in UD_n$ has $\pi_1 = n$ since $\pi_1 < \pi_2$ is required, so $|A| = 0$; the sum over $c = n$ to $n - 1$ is empty, hence 0, and the identity holds. Assume henceforth the general argument, which also covers this case.

We show F restricts to a bijection from A onto the disjoint union $\bigsqcup_{c \geq a} B_c$, with inverse G . Let $\pi \in A$. Then $\pi_1 = a$ and $\pi \in UD_n$, so $\pi_2 > \pi_1$ and Lemma 5 applies. By Lemma 3, $\sigma = F(\pi) \in DU_{n-1}$ with $c := \sigma_1 \geq a$. By Lemma 4, $\sigma_{n-1} = b^*$. By Lemma 6, $\text{Rec}(\sigma) = S^*$. Thus $\sigma \in B_c$ for $c \geq a$.

Conversely, let $\sigma \in B_c$ for some $c \geq a$, so $\sigma \in DU_{n-1}$, $\sigma_1 = c \geq a$, $\sigma_{n-1} = b^*$, $\text{Rec}(\sigma) = S^*$. Let $\pi = G(\sigma)$, so $\pi_1 = a$ and $F(\pi) = \sigma$. By Lemma 3, the DU inequalities give the tail UD inequalities for π , and $c \geq a$ gives $\pi_1 < \pi_2$, so $\pi \in UD_n$ and in particular $\pi_2 > \pi_1$. By Lemma 4, $\pi_n = b$. By Lemma 6, $\text{Rec}(\pi) = S$. Thus $\pi \in A$.

Since F, G are inverses by Lemma 2, they give a bijection between A and $\bigsqcup_{c=a}^{n-1} B_c$. Taking cardinalities gives

$$N_{ud}(n, a, b, S) = |A| = \sum_{c=a}^{n-1} |B_c| = \sum_{c=a}^{n-1} N_{du}(n - 1, c, b^*, S^*),$$

with the empty sum equal to 0 when $a = n$.

□

4. Computational verification

The search space was exhaustive enumeration. For each n with $2 \leq n \leq 8$, all $n!$ permutations of $[n]$ were enumerated, each was classified as UD_n or DU_n , π_1 and π_n and $\text{Rec}(\pi)$ were computed, $N_{ud}(n, a, b, S)$ and $N_{du}(n, a, b, S)$ were tabulated for all $a \in [n]$ and $b \in [n]$ and $S \subset [n]$, b^* and S^* were formed, both sides of the equation were evaluated, and any tuple (n, a, b, S) with $1 \in S$ and $2 \in S$ where equality fails was reported.

The falsification run exited with code 0. Sanity checks confirmed Euler totals $UD = DU = \{1 : 1, 2 : 1, 3 : 2, 4 : 5, 5 : 16, 6 : 61, 7 : 272, 8 : 1385\}$ matching expected values, hand values, and record and classifier cross-checks. No counterexample was found for $n = 2$ to 8, with 5630 cases tested.

The independent check exited with code 0 and confirmed Lemma 1 on value standardization, Lemma 2 on the deletion bijection, Lemma 3 on pattern and first value, Lemma 4 on last value, Lemma 5 on records, Lemma 6 on the shifted record set, and the theorem sum identity, with all checks passed.

Lean 4 status for Theorem 1: failed to compile. The Lean formalization exited with code 1 with error `unexpected identifier; expected command`, after 3 repairs, with `compiles: false` and `sorry_free: false`. There is therefore no machine-checked sorry-free proof. Because no Lean statement compiled, there is no back-translation faithfulness verdict to report. A sorry-free Lean proof, if it existed, would certify only the Lean statement, which would still have to be read against the informal statement.

5. Limitations and open questions

Verification is bounded to $n \leq 8$ by exhaustive enumeration and does not itself prove the theorem for larger n . The proof covers $n \geq 2$ and $b \neq a$ with $1, 2 \in S$ and treats the empty sum as 0. Complement and reversal equinumerosity, parity and modular congruences, and other refinements in the seed topic are not proved here. No `## GAP` markers remain in the verified proof to carry forward.

Novelty

The automated novelty verdict was `APPARENTLY_NEW`, with closest known results listed as Entringer recurrence for alternating permutations by deletion of the first entry and order-isomorphism standardization, and distribution of left-to-right maxima and joint maxima and minima statistics on up-down and down-up permutations.

Exact queries run were:

Queries (3): 1. recurrence number of up-down alternating permutations with given first letter last letter and left-to-right maxima set in terms of down-up permutations of length n-1 2. joint distribution of first value last value and records left-to-right maxima in alternating permutations Entringer numbers refinement 3. enumeration of alternating permutations by deletion of first entry and order-isomorphism standardization recurrence

Search terms (3): 1. Entringer numbers first last entry 2. alternating permutations left-to-right maxima record set recurrence 3. up-down permutations standardization deletion first element

Follow-up queries (2): 1. Entringer numbers joint distribution first entry last entry left-to-right maxima 2. alternating permutations deletion of first entry standardization records recurrence

Retrieval returned 27 hits in total from arXiv and Crossref, and OpenAlex searches failed with HTTP 503 Service Unavailable for all three queries, so OpenAlex contributed 0 hits. Evidence base is retrieval, not

memory-only. This is a bounded automated search over those indexes, not a literature review, and is blind to books, journals outside them, and anything phrased differently.