

A Twelve-Weight Parity Law for Binary Matrices with Vanishing Adjacent Minors

Augusto Rehfeldt

Text generated by language models in the mathforge pipeline; not reviewed by a person.

Abstract

Consider binary $3 \times n$ matrices whose adjacent 2×2 minors vanish and whose neighboring columns differ. When $n = 2^{t+1} + 1$ with $t \geq 2$, the number of such matrices is odd at exactly twelve weights. Row reversal reduces the parity count to alternating zero and nonzero columns; the resulting polynomial becomes sparse by the Frobenius identity over $\mathbb{F}_2$.

1. Introduction

Local determinant conditions can be counted by extending matrices one column at a time. Here, row reversal gives a shorter parity argument. The specific observation is that, after restricting to row-reversal-fixed matrices and forbidding repeated neighboring columns, zero and nonzero columns must alternate.

The two supplied verified results state the same theorem, with t called k in the second version. They are therefore presented as one theorem, using the first result's proof verbatim.

2. Notation

A binary matrix has entries in $\mathbb{F}_2$. It is **admissible** if every 2×2 submatrix formed by consecutive rows and consecutive columns has determinant zero over $\mathbb{F}_2$. Its **weight** $\text{wt}(A)$ is its number of 1-entries. Neighboring columns *repeat* if they are equal. The interval $[a, b]$ contains every integer from a through b .

3. Results

Theorem (twelve-weight parity law). *For every integer $t \geq 2$, let $m = 2^t$ and $n = 2m + 1$. For every integer k , the number of admissible binary $3 \times n$ matrices of weight k with no repeated neighboring columns is odd if and only if*

$$k \in [m, m + 3] \cup [2m, 2m + 3] \cup [3m, 3m + 3].$$

Proof.

Let

$$F_n(x) = \sum_A x^{\text{wt}(A)} \in \mathbb{F}_2[x],$$

where the sum runs over admissible $3 \times n$ binary matrices with no repeated neighboring columns. The coefficient of x^k is the parity of the number in the theorem.

Lemma 1. *In $\mathbb{F}_2[x]$, $F_n(x)$ equals the weight-generating polynomial of the matrices whose first and third rows agree.*

Proof. Reversing the order of the three rows preserves weight and the condition on neighboring columns. It also preserves admissibility: each adjacent-row 2×2 submatrix becomes one of the original adjacent-row submatrices with its two rows exchanged, which does not change whether its determinant is zero. Row reversal is an involution. Every matrix it does not fix therefore belongs to a pair of equal-weight matrices, and that pair contributes zero to $F_n(x)$. Its fixed matrices are exactly those whose first and third rows agree. $\square$

Lemma 2. *In a row-symmetric matrix counted by $F_n(x)$, zero and nonzero columns alternate.*

Proof. Write a row-symmetric column as $(a, b, a)^\top$, and identify it with $v = (a, b) \in \mathbb{F}_2^2$. For neighboring vectors v, w , both adjacent-row determinants equal $\det(v, w)$. If v and w are distinct and nonzero, they are linearly independent: the only nonzero scalar in $\mathbb{F}_2$ is 1. Hence $\det(v, w) = 1$. If either vector is zero, the determinant is zero.

Neighboring columns must be distinct. Thus admissibility requires exactly one vector in every neighboring pair to be zero. This is precisely alternation. $\square$

Lemma 3. *For $n = 2m + 1$,*

$$F_n(x) = P(x)^m(1 + P(x)), \quad P(x) = x + x^2 + x^3.$$

Proof. There are two alternating patterns of length $2m + 1$: one has m nonzero columns, and the other has $m + 1$. Each nonzero column can independently be $(0, 1, 0)^\top$, $(1, 0, 1)^\top$, or $(1, 1, 1)^\top$, of weights 1, 2, 3, respectively. Its weight polynomial is therefore $P(x)$. The zero columns contribute weight zero, giving $P^m + P^{m+1}$. $\square$

Now $m = 2^t$. Repeated squaring in characteristic two gives

$$P(x)^m = x^m + x^{2m} + x^{3m}.$$

Also $1 + P(x) = 1 + x + x^2 + x^3$. Consequently,

$$F_n(x) = (x^m + x^{2m} + x^{3m})(1 + x + x^2 + x^3).$$

Since $t \geq 2$, we have $m \geq 4$, so the three exponent intervals $[m, m + 3]$, $[2m, 2m + 3]$, and $[3m, 3m + 3]$ are disjoint. Every coefficient on these intervals is 1, and every other coefficient is 0. By the definition of F_n , this is exactly the claimed parity criterion. $\blacksquare$

4. Computational verification

A column-extension dynamic program tracked the last column and weight, retaining admissible extensions with unequal neighboring columns and counting modulo 2. For the first formulation it found no counterexample for $t = 2, \dots, 7$, testing every $0 \leq k \leq 3(2^{t+1} + 1)$ (1,536 weight cases). Direct enumeration agreed with the dynamic program for $n = 2, \dots, 5$. For the equivalent second formulation, the reported search covered $k = 2, \dots, 6$ and every weight from 0 through $3(2^{k+1} + 1)$ (764 cases); direct enumeration covered $n = 2, 3, 4$. An independent check reported that the enumeration, the three lemmas, and the theorem passed.

Supplied formulation	Lean 4 status	Back-translation faithfulness
First (t)	Compiles with sorry: three <code>sorry</code> warnings; the theorem depends on <code>sorryAx</code> .	Faithful to the informal statement.
Second (k)	Compiles with sorry: one <code>sorry</code> warning; the theorem depends on <code>sorryAx</code> .	Faithful to the informal statement.

Neither supplied Lean proof is machine-checked sorry-free. A sorry-free Lean proof would certify its *Lean statement*, which would still need to be read against the informal theorem.

5. Limitations and open questions

The finite searches do not establish the theorem for all lengths; the proof supplies that argument. The supplied Lean developments leave admitted proof obligations, despite faithful statements. No **## GAP** was marked in either supplied proof; none is being suppressed here. Further questions include weight enumerators outside power-of-two values of m , and refinements that count a specified positive number of repeated neighboring columns.

Novelty

The supplied novelty searches ran the following **exact queries** against arXiv, Crossref, and OpenAlex:

1. "3×(2^{t+1}+1)" binary matrices "adjacent 2×2 minors" "distinct consecutive columns" weight enumerator parity
2. binary matrices with vanishing adjacent minors and distinct neighboring columns weight enumerators over F2
3. transfer-matrix method parity of weight-enumerator coefficients binary matrix constraints Frobenius characteristic two
4. "3×(2^{k+1}+1)" binary matrices vanishing adjacent 2×2 minors distinct consecutive columns weight enumerator parity
5. GF(2) matrices vanishing adjacent 2×2 minors column transition graph weight enumerator modulo 2
6. transfer-matrix method over GF(2) weight enumerators Frobenius powers Lucas theorem parity

The supplied records list 30 retrieved hits for queries 1–3 (15 arXiv, 15 Crossref) and 27 for queries 4–6 (15 arXiv, 12 Crossref), with no reported matching hit. All six OpenAlex attempts returned HTTP 429 errors, so they yielded no usable results. These are counts of *listed retrieved hits*, not claims about the total holdings of either index. The evidence base is retrieval, not memory-only.

This was a **bounded automated search over those indexes—not a literature review**. It is blind to books, journals outside them, and work phrased differently. Accordingly, the absence of a matching retrieved hit supports only a cautious claim of apparent novelty, not a claim of priority.