

# 3-Term Progression-Free Subsets of $[n]$ by Size and Sum: A Low-Sum Recurrence and a Central-Sum Parity

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## Abstract

This paper proves two verified enumerative statements about 3-term arithmetic progression-free subsets of  $[n]$ , refined by cardinality and element sum. First, small-sum progression-free sets satisfy Pascal's recurrence exactly under an explicit inequality. Second, for odd intervals and odd sizes, the number of progression-free sets with reflection-central sum is even. Both statements were checked by exhaustive enumeration for small  $n$ , independently rechecked, and formalized as Lean 4 statements that compile with `sorry`.

## 1. Introduction

Let  $A(n, k, s)$  denote the number of 3-term arithmetic progression-free subsets of  $[n]$  with exactly  $k$  elements and element sum exactly  $s$ . The literature on progression-free sets emphasizes maximal size, total counts, asymptotics, and generating functions. Endpoint recurrences conditioning on the largest element are usually stated as inequalities, because adjoining the maximum can create a forbidden progression and therefore breaks Pascal's identity.

What is new here is two exact doubly-refined regimes. Theorem 1 gives an explicit low-sum threshold,  $2s \leq 3n + (k-1)(k-2)$ , under which adjoining  $n$  cannot create a 3-term progression, so  $A(n, k, s) = A(n-1, k, s) + A(n-1, k-1, s-n)$  holds exactly. Theorem 2 gives a central-sum parity vanishing result: for  $n = 2m+1$ ,  $k = 2t+1$ , and  $s = k(m+1)$ , the corresponding count is even, because reflection is a fixed-point-free involution and any symmetric odd set must contain a progression through the centre. The novelty claims for these specific combined cardinality-and-sum statements are supported only by a bounded automated search described in Section 5, not by a full literature review.

## 2. Notation

For  $n \geq 1$ ,  $[n]$  denotes  $\{1, 2, \dots, n\}$  and  $[0]$  denotes the empty set. A set  $S$  is 3-term arithmetic progression-free if it contains no three distinct elements  $x, y, z$  with  $x+z = 2y$ , equivalently no set of the form  $\{a, a+d, a+2d\}$  with  $d > 0$ . The cardinality of  $S$  is  $|S|$  and  $\text{sum}(S)$  is the sum of its elements, with  $\text{sum}([0]) = 0$ . The symbol  $A(n, k, s)$  denotes the number of subsets  $S$  of  $[n]$  such that  $S$  is 3-term arithmetic progression-free,  $|S| = k$ , and  $\text{sum}(S) = s$ , with  $A(0, 0, 0) = 1$  and  $A(0, k, s) = 0$  otherwise. The term  $A(n-1, k-1, s-n)$  is understood as 0 when  $s-n$  is negative. For Theorem 2,  $m$  is the half-interval parameter,  $t$  parametrizes odd size  $k \geq 3$ ,  $n = 2m+1$  is the odd interval length,  $c = m+1$  is the centre,  $r(x) = n+1-x$  is reflection, and  $s = k(m+1) = k(n+1)/2$  is the reflection-central sum.

## 3. Results

**Theorem 1** (Low-Sum Exact Pascal Recurrence for 3-AP-Free Sets by Size and Sum). *For every integer  $n \geq 1$ , every integer  $k \geq 2$ , and every integer  $s$  satisfying*

$$2s \leq 3n + (k-1)(k-2),$$

let  $A(n, k, s)$  be the number of subsets  $S \subseteq [n]$  such that  $S$  is 3-term arithmetic progression-free,  $|S| = k$ , and  $\text{sum}(S) = s$ . Then

$$A(n, k, s) = A(n-1, k, s) + A(n-1, k-1, s-n),$$

where  $A(n-1, k-1, s-n)$  is understood as 0 if  $s-n < 0$ .

Here  $[n] = \{1, \dots, n\}$ ,  $[0] = \emptyset$ , and  $S$  is 3-term arithmetic progression-free if there do not exist pairwise distinct  $x, y, z \in S$  with  $x + z = 2y$ .

**Lemma 1.** If  $|S| \leq 2$  then  $S$  is 3-term arithmetic progression-free.

*Proof.* A 3-term progression in  $S$  requires three pairwise distinct elements  $x, y, z \in S$  with  $x + z = 2y$ . If  $|S| \leq 2$ , such a triple does not exist. Hence the defining forbidden configuration is absent.  $\square$

**Lemma 2.** If  $S$  is 3-term arithmetic progression-free and  $T \subseteq S$ , then  $T$  is 3-term arithmetic progression-free.

*Proof.* Suppose  $T$  were not 3-term arithmetic progression-free. Then there exist pairwise distinct  $x, y, z \in T$  with  $x + z = 2y$ . Since  $T \subseteq S$ , we have  $x, y, z \in S$ , so  $S$  contains such a triple and is not 3-term arithmetic progression-free. By contraposition, if  $S$  is 3-term arithmetic progression-free then so is every  $T \subseteq S$ .  $\square$

**Lemma 3.** Let  $R$  be a set of  $r$  distinct positive integers. Then  $\text{sum}(R) \geq r(r+1)/2$ .

*Proof.* Write the elements in increasing order  $a_1 < a_2 < \dots < a_r$ , where each  $a_i$  is a positive integer. Then  $a_1 \geq 1$ . For each  $i$ , since  $a_{i+1}$  and  $a_i$  are integers with  $a_{i+1} > a_i$ , we have  $a_{i+1} \geq a_i + 1$ . By induction,  $a_i \geq i$  for all  $i$ : the case  $i = 1$  holds, and if  $a_i \geq i$  then  $a_{i+1} \geq a_i + 1 \geq i + 1$ . Therefore

$$\text{sum}(R) = \sum_{i=1}^r a_i \geq \sum_{i=1}^r i = \frac{r(r+1)}{2}.$$

$\square$

**Lemma 4.** Let  $S \subseteq [n]$  with  $n \in S$ . Suppose  $S$  contains pairwise distinct  $a, b, c \in S$  with  $a + c = 2b$  and with at least one of  $a, b, c$  equal to  $n$ . Then, after possibly interchanging the names of the two summands, the progression has the form  $x + n = 2y$  for pairwise distinct  $x, y \in S \setminus \{n\}$ .

*Proof.* The three elements are pairwise distinct and one of them equals  $n$ . Consider the three cases.

If  $b = n$ , then  $a + c = 2n$  with  $a, c \in S$  distinct from  $n$  and from each other. Since  $a, c \leq n-1$ , we obtain  $a + c \leq 2n-2 < 2n$ , contradicting  $a + c = 2n$ . Hence  $b \neq n$ .

Thus  $n$  equals  $a$  or  $c$ . If  $a = n$ , then  $n + c = 2b$ ; set  $x = c$  and  $y = b$ . If  $c = n$ , then  $a + n = 2b$ ; set  $x = a$  and  $y = b$ . In either case  $x + n = 2y$ . Since  $a, b, c$  are pairwise distinct,  $x$  and  $y$  are distinct from  $n$  and from each other. Since  $x, y \in S$  and  $x, y \neq n$ , we have  $x, y \in S \setminus \{n\}$ .  $\square$

**Lemma 5.** Assume  $n \geq 1$ ,  $k \geq 2$ , and  $2s \leq 3n + (k-1)(k-2)$ . Let  $T \subseteq [n-1]$  with  $|T| = k-1$  and  $\text{sum}(T) = s-n$  (so in particular  $s-n \geq 0$ ). Let  $m = \max(T)$ . Then  $m \leq n/2$ .

*Proof.* Since  $k \geq 2$ , we have  $|T| = k - 1 \geq 1$ , so  $T$  is nonempty and  $m$  exists. The set  $T \setminus \{m\}$  consists of  $k - 2$  distinct positive integers. By Lemma 3 applied with  $r = k - 2$ ,

$$\text{sum}(T \setminus \{m\}) \geq \frac{(k-2)(k-1)}{2}.$$

Since  $\text{sum}(T) = m + \text{sum}(T \setminus \{m\})$  and  $\text{sum}(T) = s - n$ ,

$$s - n \geq m + \frac{(k-1)(k-2)}{2},$$

so

$$m \leq (s - n) - \frac{(k-1)(k-2)}{2}.$$

Multiplying by 2 gives  $2m \leq 2s - 2n - (k-1)(k-2)$ . By hypothesis  $2s \leq 3n + (k-1)(k-2)$ , hence

$$2m \leq (3n + (k-1)(k-2)) - 2n - (k-1)(k-2) = n.$$

Therefore  $2m \leq n$ , i.e.  $m \leq n/2$ .

□

**Lemma 6.** Assume  $n \geq 1$ ,  $k \geq 2$ , and  $2s \leq 3n + (k-1)(k-2)$ . Let  $\mathcal{F}_0$  be the family of 3-term arithmetic progression-free sets  $S \subseteq [n]$  with  $|S| = k$ ,  $\text{sum}(S) = s$ , and  $n \notin S$ . Let  $\mathcal{F}_1$  be the family of such sets with  $n \in S$ . Then  $|\mathcal{F}_0| = A(n-1, k, s)$  and  $|\mathcal{F}_1| = A(n-1, k-1, s-n)$ .

*Proof.* A subset  $S \subseteq [n]$  satisfies  $n \notin S$  if and only if  $S \subseteq [n-1]$ . The properties  $|S| = k$ ,  $\text{sum}(S) = s$ , and being 3-term arithmetic progression-free depend only on  $S$  itself. Hence  $\mathcal{F}_0$  coincides exactly with the family counted by  $A(n-1, k, s)$ , giving  $|\mathcal{F}_0| = A(n-1, k, s)$ .

Consider  $\mathcal{F}_1$ . If  $s - n < 0$ , then no  $S \ni n$  has sum  $s$ , because every  $S \subseteq [n]$  containing  $n$  has  $\text{sum}(S) \geq n > s$ . Thus  $\mathcal{F}_1 = \emptyset$ . Also by convention  $A(n-1, k-1, s-n) = 0$ . Hence  $|\mathcal{F}_1| = A(n-1, k-1, s-n)$  in this case.

Assume henceforth  $s - n \geq 0$ . Define  $\Phi(S) = S \setminus \{n\}$  and  $\Psi(T) = T \cup \{n\}$ . For arbitrary (not necessarily progression-free) sets,  $\Phi$  is a bijection between  $\{S \subseteq [n] : n \in S, |S| = k, \text{sum}(S) = s\}$  and  $\{T \subseteq [n-1] : |T| = k-1, \text{sum}(T) = s-n\}$ , with inverse  $\Psi$ , because removing  $n$  reduces cardinality by 1 and sum by  $n$ , while adding  $n$  to  $T \subseteq [n-1]$  increases them by the same amounts.

We show  $\Phi$  restricts to a bijection between  $\mathcal{F}_1$  and the family  $\mathcal{G}$  counted by  $A(n-1, k-1, s-n)$ , namely 3-term arithmetic progression-free  $T \subseteq [n-1]$  with  $|T| = k-1$  and sum  $s-n$ .

If  $S \in \mathcal{F}_1$ , then  $T = \Phi(S)$  satisfies  $T \subseteq [n-1]$ ,  $|T| = k-1$ ,  $\text{sum}(T) = s-n$ , and  $T \subseteq S$ ; by Lemma 2,  $T$  is 3-term arithmetic progression-free. Thus  $\Phi(S) \in \mathcal{G}$ .

Conversely, let  $T \in \mathcal{G}$  and  $S = \Psi(T) = T \cup \{n\}$ . Then  $S \subseteq [n]$ ,  $|S| = k$ ,  $\text{sum}(S) = s$ . It remains to show  $S$  is 3-term arithmetic progression-free. If  $k = 2$ , then  $|S| = 2$  and  $S$  is 3-term arithmetic progression-free by Lemma 1. Suppose  $k \geq 3$  and, for contradiction, that  $S$  is not 3-term arithmetic progression-free. Then  $S$  contains pairwise distinct  $a, b, c$  with  $a + c = 2b$ . If none of  $a, b, c$  equals  $n$ , then  $a, b, c \in T$ , contradicting the fact that  $T$  is 3-term arithmetic progression-free. Hence at least one equals  $n$ , and by Lemma 4 there exist pairwise distinct  $x, y \in T$  with  $x + n = 2y$ .

But  $T \subseteq [n-1]$ , so  $x \geq 1$ , and  $y \leq \max(T) \leq n/2$  by Lemma 5. Consequently  $2y \leq n < n+1 \leq n+x$ , so  $2y \neq x+n$ , a contradiction. Hence no such triple exists in  $S$ , and  $S$  is 3-term arithmetic progression-free. Thus  $\Psi(T) \in \mathcal{F}_1$ .

Therefore  $\Phi$  and  $\Psi$  are mutually inverse bijections between  $\mathcal{F}_1$  and  $\mathcal{G}$ , so  $|\mathcal{F}_1| = |\mathcal{G}| = A(n-1, k-1, s-n)$ .

□

*Proof of the theorem.* Fix  $n \geq 1$ ,  $k \geq 2$ , and  $s$  with  $2s \leq 3n + (k-1)(k-2)$ . Let  $\mathcal{F}$  be the family of subsets  $S \subseteq [n]$  that are 3-term arithmetic progression-free with  $|S| = k$  and  $\text{sum}(S) = s$ , so  $|\mathcal{F}| = A(n, k, s)$ . Every  $S \in \mathcal{F}$  either contains  $n$  or does not. With  $\mathcal{F}_0, \mathcal{F}_1$  as in Lemma 6,  $\mathcal{F}$  is the disjoint union of  $\mathcal{F}_0$  and  $\mathcal{F}_1$ . Hence

$$A(n, k, s) = |\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1|.$$

By Lemma 6,  $|\mathcal{F}_0| = A(n-1, k, s)$  and  $|\mathcal{F}_1| = A(n-1, k-1, s-n)$ . Substituting gives

$$A(n, k, s) = A(n-1, k, s) + A(n-1, k-1, s-n),$$

as claimed. □

**Theorem 2** (Central-Sum Parity Vanishing for Odd-Size 3-AP-Free Sets in Odd Intervals). *For every pair of integers  $m \geq 1$  and  $t \geq 1$ , let  $n = 2m+1$  and  $k = 2t+1$  and  $s = k(m+1)$ . Then the number of subsets  $S$  of  $[n] = \{1, 2, \dots, n\}$  such that  $S$  contains no 3-term arithmetic progression and  $S$  has exactly  $k$  elements with sum exactly  $s$  is even.*

A 3-term arithmetic progression is a set  $\{a, a+d, a+2d\}$  with  $d > 0$ .

Let  $m \geq 1$ ,  $t \geq 1$  be fixed throughout, and define  $n = 2m+1$ ,  $k = 2t+1$ ,  $s = k(m+1)$ , and  $c = m+1$ . Note that  $c = (n+1)/2$ , that  $n+1 = 2c$ , and that  $s = k(n+1)/2$ . Note also that  $k \geq 3$ . Let

$$\mathcal{F} = \{S \subseteq [n] : |S| = k, \sum_{x \in S} x = s, S \text{ contains no 3-term arithmetic progression}\}.$$

Since  $[n]$  is finite,  $\mathcal{F}$  is finite. If  $k > n$  then  $\mathcal{F}$  is empty and  $|\mathcal{F}| = 0$ , which is even. The argument below covers all cases including this one.

Define  $r : [n] \rightarrow [n]$  by  $r(x) = n+1-x$ , and  $\Phi : \mathcal{P}([n]) \rightarrow \mathcal{P}([n])$  by  $\Phi(S) = \{r(x) : x \in S\}$ .

**Lemma 1.** *The map  $r$  is a bijection of  $[n]$  onto  $[n]$ , satisfies  $r(r(x)) = x$  for all  $x \in [n]$ , satisfies  $r(c) = c$ , and satisfies  $r(x) \neq x$  for  $x \neq c$ .*

*Proof.* If  $1 \leq x \leq n$  then  $n+1-x$  is an integer, and  $x \leq n$  implies  $n+1-x \geq 1$  while  $x \geq 1$  implies  $n+1-x \leq n$ , so  $r(x) \in [n]$ . For any  $x$ ,  $r(r(x)) = n+1-(n+1-x) = x$ , so  $r$  is its own inverse and hence a bijection. Since  $n+1 = 2m+2$  and  $c = m+1$ ,  $r(c) = (2m+2) - (m+1) = m+1 = c$ . If  $r(x) = x$  then  $n+1-x = x$ , so  $2x = n+1 = 2c$ , so  $x = c$ . By contraposition,  $x \neq c$  implies  $r(x) \neq x$ . □

**Lemma 2.** *The map  $\Phi$  satisfies  $\Phi(\Phi(S)) = S$  for all  $S \subseteq [n]$ , and  $|\Phi(S)| = |S|$  for all  $S$ .*

*Proof.* By definition  $\Phi(S) = r(S)$ , the image of  $S$  under  $r$ . Since  $r$  is a bijection of  $[n]$ , it restricts to a bijection from  $S$  onto  $r(S)$ , so  $|r(S)| = |S|$ . Applying  $r$  twice is the identity by Lemma 1, so  $r(r(S)) = S$ . □

**Lemma 3.** *For every  $S \subseteq [n]$ ,  $\sum_{y \in \Phi(S)} y = |S|(n+1) - \sum_{x \in S} x$ . In particular, if  $|S| = k$  and  $\sum_{x \in S} x = s$ , then  $\sum_{y \in \Phi(S)} y = s$ .*

*Proof.* Since  $r$  is injective, summation over  $y \in \Phi(S)$  is summation over  $r(x)$  for  $x \in S$ , each appearing once. Hence

$$\sum_{y \in \Phi(S)} y = \sum_{x \in S} (n+1-x) = |S|(n+1) - \sum_{x \in S} x.$$

Suppose  $|S| = k$  and  $\sum_S = s = k(m+1)$ . Since  $n+1 = 2(m+1)$ ,

$$|S|(n+1) - \sum_S = k \cdot 2(m+1) - k(m+1) = k(m+1) = s.$$

Thus  $\Phi(S)$  has the same cardinality  $k$  and the same sum  $s$ .

□

**Lemma 4.** *For every  $S \subseteq [n]$ ,  $S$  contains a 3-term arithmetic progression if and only if  $\Phi(S)$  does. Hence  $S$  is 3-term-progression-free if and only if  $\Phi(S)$  is.*

*Proof.* Suppose  $S$  contains  $\{a, a+d, a+2d\}$  for some  $a \geq 1$  and  $d > 0$ , with all three elements in  $S$ . Their images under  $r$  are in  $\Phi(S)$  by definition. Write  $a' = r(a+2d) = n+1-a-2d$ . Then

$$a' + d = n+1-a-d = r(a+d), \quad a' + 2d = n+1-a = r(a).$$

Thus  $\{a', a'+d, a'+2d\} = \{r(a+2d), r(a+d), r(a)\} \subseteq \Phi(S)$ , with the same  $d > 0$ . So  $\Phi(S)$  contains a 3-term arithmetic progression. Applying the same implication to  $\Phi(S)$  and using  $\Phi(\Phi(S)) = S$  from Lemma 2 gives the converse: if  $\Phi(S)$  contains a 3-term progression then so does  $\Phi(\Phi(S)) = S$ . The statement about progression-free sets follows by negation.

□

**Lemma 5.** *If  $S \subseteq [n]$  satisfies  $\Phi(S) = S$  and  $|S|$  is odd, then  $c \in S$ .*

*Proof.* Assume  $\Phi(S) = S$ , i.e.  $r(S) = S$ . Let  $S' = S \setminus \{c\}$ . For  $x \neq c$ , Lemma 1 gives  $r(x) \neq x$  and  $r(x) \neq c$ , where the latter holds because  $r$  is bijective and  $r(c) = c$ . Hence  $r$  maps  $[n] \setminus \{c\}$  to itself without fixed points, and maps  $S'$  to itself: if  $x \in S'$  then  $r(x) \in S$  by symmetry and  $r(x) \neq c$ , so  $r(x) \in S'$ .

The set  $[n] \setminus \{c\}$  is the disjoint union of the  $m$  pairs  $\{x, r(x)\}$  with  $1 \leq x < c$ . Indeed, for each  $x < c$ ,  $r(x) = n+1-x > n+1-c = c$ , so each such pair has one element below  $c$  and one above  $c$ , distinct by Lemma 1; every element of  $[n] \setminus \{c\}$  lies in exactly one such pair. Since  $S'$  is invariant under  $r$ , for each such pair  $S'$  contains either both elements or neither: if  $x \in S'$  then  $r(x) \in S'$ , and if  $r(x) \in S'$  then  $x = r(r(x)) \in S'$ . Therefore  $S'$  is a union of some subcollection of these 2-element pairs, and  $|S'| = 2q$  for some integer  $q \geq 0$ , hence  $|S'|$  is even.

Now  $|S| = |S'| + |S \cap \{c\}|$ , where  $|S \cap \{c\}|$  is 1 if  $c \in S$  and 0 otherwise. If  $|S|$  is odd and  $|S'|$  is even,  $|S \cap \{c\}|$  must equal 1. Hence  $c \in S$ .

□

**Lemma 6.** *If  $S \subseteq [n]$  satisfies  $\Phi(S) = S$  and contains  $c$  and some  $a \neq c$ , then  $S$  contains a 3-term arithmetic progression.*

*Proof.* Let  $a \in S$  with  $a \neq c$ . By symmetry  $r(a) = n+1-a \in S$ , and by Lemma 1  $r(a) \neq a$ . Also  $r(a) \neq c$  since  $a \neq c$  and  $r$  is bijective with  $r(c) = c$ . So  $a, c, r(a)$  are three distinct elements of  $S$ . We show they form an arithmetic progression with middle term  $c$ .

Since  $n+1 = 2c$ , we have  $a+r(a) = a+(n+1-a) = n+1 = 2c$ . If  $a < c$  then  $r(a) = 2c-a > c$ . Setting  $d = c-a > 0$  gives  $a+d = c$  and  $a+2d = 2c-a = r(a)$ , so  $\{a, c, r(a)\} = \{a, a+d, a+2d\}$ . If  $a > c$  then  $r(a) < c$ ; interchanging the roles of  $a$  and  $r(a)$ , with  $d = c-r(a) > 0$ , gives  $\{r(a), c, a\} = \{r(a), r(a)+d, r(a)+2d\}$ . In either case the three distinct elements of  $S$  constitute a set of the form  $\{b, b+d, b+2d\}$  with  $d > 0$ .

□

**Lemma 7.** *Let  $\mathcal{G}$  be a finite set and  $\Psi : \mathcal{G} \rightarrow \mathcal{G}$  an involution, i.e.  $\Psi(\Psi(z)) = z$  for all  $z$ , with no fixed point, i.e.  $\Psi(z) \neq z$  for all  $z$ . Then  $|\mathcal{G}|$  is even.*

*Proof.* Define a relation on  $\mathcal{G}$  by  $z \sim w$  if  $w = z$  or  $w = \Psi(z)$ . This is an equivalence relation: it is reflexive by definition; it is symmetric because  $w = \Psi(z)$  implies  $z = \Psi(w)$  by the involution property; it is transitive because if  $y \in \{z, \Psi(z)\}$  and  $w \in \{y, \Psi(y)\}$  then, using  $\Psi(\Psi(z)) = z$ ,  $w \in \{z, \Psi(z)\}$ . Each equivalence class is  $\{z, \Psi(z)\}$ , which has exactly two elements since  $\Psi(z) \neq z$ . Distinct classes are disjoint and their union is  $\mathcal{G}$ . Since  $\mathcal{G}$  is finite, it is a finite disjoint union of 2-element sets, so its cardinality is twice the number of classes, hence even. □

*Proof of the Theorem.* Lemmas 2, 3, and 4 show that  $\Phi$  maps  $\mathcal{F}$  to itself: if  $S \in \mathcal{F}$  then  $|\Phi(S)| = k$ ,  $\sum_{\Phi(S)} = s$ , and  $\Phi(S)$  is progression-free, so  $\Phi(S) \in \mathcal{F}$ . Restricted to  $\mathcal{F}$ ,  $\Phi$  remains an involution by Lemma 2.

We show this involution has no fixed point in  $\mathcal{F}$ . Suppose, for contradiction, that  $S \in \mathcal{F}$  satisfies  $\Phi(S) = S$ . Then  $|S| = k = 2t + 1$  is odd, so Lemma 5 gives  $c \in S$ . Since  $k = 2t + 1 \geq 3$ ,  $S$  has at least 3 elements and hence contains some  $a \neq c$ . Lemma 6 then implies  $S$  contains a 3-term arithmetic progression, contradicting the definition of  $\mathcal{F}$ . Therefore  $\Phi(S) \neq S$  for every  $S \in \mathcal{F}$ .

Since  $\mathcal{F}$  is finite and carries the fixed-point-free involution  $\Phi$ , Lemma 7 yields that  $|\mathcal{F}|$  is even. That is, the number of 3-term-progression-free subsets  $S \subseteq [n]$  with  $|S| = k$  and sum  $s$  is even. □

## 4. Computational Verification

For Theorem 1, all subsets of  $[n]$  were enumerated for each  $n$  from 1 to 12, each  $k$  from 0 to  $n$ , and each  $s$  from 0 to  $n(n+1)/2$ , with 3-term progression-freeness, cardinality, and sum tested directly. The claimed equality was then checked for all triples with  $k \geq 2$  and  $2s \leq 3n + (k-1)(k-2)$ . A total of 1624 cases satisfying the hypotheses were tested, and no counterexample was found. Sanity checks on 14 known values and an independent cross-check at  $n = 5$  also passed. A separate independent program rechecked Lemma 1 through Lemma 6 and the final recurrence, and all checks passed.

For Theorem 2, for each odd  $n$  in  $\{3, 5, 7, 9, 11, 13\}$  and each odd  $k$  with  $3 \leq k \leq n$ , with  $s = k(n+1)/2$ , all  $2^n$  subsets of  $[n]$  were enumerated and tested for progression-freeness, cardinality  $k$ , and sum  $s$ , and the resulting count was checked to be even. A total of 21 cases were tested, and no counterexample was found. Sanity checks on known progression cases and totals passed. A separate independent program rechecked Lemma 1 through Lemma 7 and the final parity claim, and all checks passed.

The Lean 4 status is as follows. Theorem 1 compiles with `sorry`: the Lean file compiles successfully but contains 6 occurrences of `sorry`, so it is not machine-checked sorry-free. Its back-translation was judged faithful, meaning the Lean statement quantifies over the same  $n$ ,  $k$ , and  $s$  with the same inequality and counting definition, using `Finset.Icc 1 n` for  $[n]$  and truncated natural subtraction that agrees with the informal statement under the stated bounds. Theorem 2 compiles with `sorry`: the Lean file compiles successfully but contains 15 occurrences of `sorry`, so it is not machine-checked sorry-free. Its back-translation was judged faithful, meaning the Lean quantifiers, interval, cardinality condition, sum condition, progression-free condition, and evenness conclusion match the informal claim. A sorry-free Lean proof would certify only the Lean statement as written, which must still be read against the informal statement, and the above faithfulness verdicts record that comparison.

## 5. Limitations and Open Questions

Exhaustive search reaches only  $n \leq 12$  for Theorem 1 and  $n \leq 13$  for Theorem 2, so it cannot exclude large- $n$  phenomena. The low-sum bound is sufficient but its sharpness was illustrated only by the example  $n = 5$ ,  $k = 3$ , where the identity holds at  $s = 8$  and fails at  $s = 9$ . Open questions include whether the threshold can be raised for special  $k$ , whether an analogous high-sum or two-sided exact regime exists, and whether the parity argument extends to longer progressions or other symmetric sum fibres. There are no remaining

## GAP markers carried forward from the verification records, and the referees reported no issues, but that does not remove the limits of finite checking and of `sorry`-containing formalizations.

## Novelty

Automated novelty searches were run against arXiv, Crossref, and OpenAlex. This was a bounded keyword search over those indexes only, not a literature review, and it is blind to books, journals outside those indexes, preprints phrased differently, and any result using different terminology.

For Theorem 1, the exact queries were "3-AP-free"  $k$ -subsets of  $[n]$  with given sum recurrence for  $A(n,k,s)$  conditioning on largest element, enumeration of arithmetic progression-free subsets of integers by cardinality and sum bivariate generating function  $q$ -binomial analogue, and extension lemma adjoining maximum element to 3-term-AP-free set creates no 3-term progression under sum bound largest-element deletion induction. Together they returned 34 records, all unrelated to the claimed identity, concerning subjects such as  $B$  meson decays, detector performance, neutrinos, sum-free sets, complete progression-free sets, and unrelated combinatorics. Two OpenAlex calls in this round failed with HTTP 429 Too Many Requests. The evidence base is retrieval, and the verdict recorded was apparently new.

For Theorem 2, the exact queries were parity of number of 3-term arithmetic progression-free  $k$ -subsets of  $[n]$  with fixed cardinality and fixed sum central mean reflection involution, enumeration of arithmetic progression-free subsets of  $[n]$  by cardinality and sum, and fixed-point-free reflection involution  $i$  to  $n+1-i$  parity argument centrally symmetric sets contain a 3-term arithmetic progression. Together they returned 27 records, all unrelated to the claimed parity, concerning subjects such as gamma-ray bursts, exoplanets, fixed-point-free involutions in other settings, subset sums, and progression asymptotics. Three OpenAlex calls in this round failed with HTTP 429 Too Many Requests. The evidence base is retrieval, and the verdict recorded was apparently new.

## Models

### Low-Sum Exact Pascal Recurrence for 3-AP-Free Sets by Size and Sum (verified)

- Proposed the claim: muse-spark-1.3-contributor-free
- Counterexample search: muse-spark-1.3-contributor-free
- Wrote the proof: muse-spark-1.3-contributor-free
- Referee: muse-spark-1.3-contributor-free
- Independent check script: muse-spark-1.3-contributor-free
- Lean formalization: muse-spark-1.3-contributor-free
- Faithfulness judge: muse-spark-1.3-contributor-free

### Central-sum parity vanishing for odd-size 3-AP-free sets in odd intervals (verified)

- Proposed the claim: muse-spark-1.3-contributor-free
- Counterexample search: muse-spark-1.3-contributor-free
- Wrote the proof: muse-spark-1.3-contributor-free
- Referee: muse-spark-1.3-contributor-free
- Independent check script: muse-spark-1.3-contributor-free
- Lean formalization: muse-spark-1.3-contributor-free
- Faithfulness judge: muse-spark-1.3-contributor-free