Rationality of the center of a generic division algebra with a group action over a field of characteristic 0

Zhengyao Wu
School of Mathematics and Computational Science,
Xiangtan University, Xiangtan, 411105, Hunan, China
wuzhengyao@xtu.edu.cn

June 17, 2026

Contents

Abstract

Let F be a field of characteristic 0 and let $H = C_3 \times C_3$ act regularly as a transitive subgroup of S_9. We resolve Problem 5.2 of Auel–Brussel–Garibaldi–Vishne: $Z_H(F, 9)$, the center of the generic division algebra of degree 9 with H-action, is *not* stably rational and *not* rational, but is retract rational over F. The non-stable-rationality is proved by computing $\mathrm{H}^2(H, M|_H) \cong C_9$ for the restricted Procesi lattice and showing that its exponent 9 is incompatible with stable permutation (all permutation lattices have H^2-exponent dividing 3). This integral cohomological obstruction is of a different nature from the classical unramified Brauer group ($\mathrm{Br}_{\mathrm{nr}}(F(H)^H/F) \cong C_3$, Proposition 5.3), which governs the Noether problem but not $Z_H(F, 9)$. The proof generalizes to all odd primes p: for $H = C_p \times C_p \subset S_{p^2}$, one obtains $\mathrm{H}^2(H, M|_H) \cong C_{p^2}$ while permutation lattices have exponent dividing p, so $Z_H(F, p^2)$ is not stably rational.

Keywords: generic division algebra, stable rationality, unramified Brauer group, integral group cohomology, flasque resolution.

MSC2020: 14E08, 20J06, 14M20.

1 Introduction

Let F be a field of characteristic 0. For an integer $n \geq 1$ and a subgroup $H \subseteq S_n$, the *generic division algebra with H-action*, denoted $UD(F, n, H)$, has center $Z_H(F, n)$. Problem 5.2 of Auel–Brussel–Garibaldi–Vishne [1, Problem 5.2] asks:

> Determine whether $Z_H(F, 9)$ is rational, stably rational, or retract rational over F, where $H = C_3 \times C_3 \subset S_9$ is a transitive subgroup.

The rationality problem for fields of the form $F(M)^G$ has a long history. Endo and Miyata [10, §1] developed the theory of quasi-permutation modules, providing the algebraic foundation for the rationality problem. Building on this, Endo–Miyata [10, Theorem 1.2] proved that $F(M)^G$ is stably rational over F if and only if M is a quasi-permutation $\mathbb{Z}[G]$-lattice, where "quasi-rational" in the original terminology of [10] is synonymous with stably rational. Voskresenskii [17, Chapter 2] gave a comprehensive treatment via the birational geometry of algebraic tori. Colliot-Thélène and Sansuc [7, §1] developed flasque resolutions and their applications to rationality. Saltman [16, Section 3] introduced retract rationality and constructed the first examples of non-rational stably rational fields.

When $H \subset S_n$ is cyclic and transitive, $Z_H(F, n)$ is rational if F contains a primitive n-th root of unity (and retract rational for general F when n is not of the form $8m$) [1, pp. 234]; see also [7, §5, Corollary 3]. Thus $H = C_3 \times C_3$ is the first case where the rationality question is genuinely open for n odd.

In this paper we give a complete answer:

Property	*Status*	*Method*
Rational	**No (char 0)**	$\mathrm{H}^2(H, M\|_H) \cong C_9$; rational $\Rightarrow$ stably rational
Stably rational	**No (char 0)**	$\mathrm{H}^2(H, M\|_H) \cong C_9$
Retract rational	**Yes (any field)**	Saltman [16, Corollary 3.13]

The main new ingredient is the computation of the second integral cohomology group of the restricted Procesi lattice $M|_H$: $\mathrm{H}^2(H, M|_H) \cong C_9$ (Lemma 4.2). The exponent 9 of this group directly obstructs stable permutation: we prove that for any permutation $\mathbb{Z}[H]$-lattice P, every element of $\mathrm{H}^2(H, P)$ has order dividing 3 (Proposition 4.4). Consequently $M|_H$ cannot be stably permutation, and by Theorem 2.6, $Z_H(F, 9)$ is not stably rational over F (Theorem 4.7). This integral cohomological obstruction appears to be new for multiplicative invariant fields.

The logical chain is:

$$
\begin{aligned}
&\mathrm{H}^2(H, M|_H) \cong C_9 \\
&\quad\Longrightarrow\ M|_H \text{ is not stably permutation} \qquad \text{(Proposition 4.5)} \\
&\quad\Longrightarrow\ Z_H(F, 9) \text{ is not stably rational} \qquad \text{(Theorem 2.6)} \\
&\quad\Longrightarrow\ Z_H(F, 9) \text{ is not rational} \qquad \text{(rationality } \Rightarrow \text{ stable rationality).}
\end{aligned}
$$

Thus both non-stable-rationality and non-rationality hold over any field of characteristic 0 (via Theorem 2.6), without reliance on the open converse Endo–Miyata implication.

As complementary evidence, we compute $\mathrm{Ext}^1_{\mathbb{Z}[H]}(I[H], M|_H) \cong C_9 \neq C_3 \times C_3 \cong \mathrm{Ext}^1_{\mathbb{Z}[H]}(I[H], 8 \cdot \mathbb{Z}[H] \oplus \mathbb{Z})$ (Proposition 4.12), which provides an independent verification that $M|_H$ is not a permutation lattice.

Finally, we prove that $Z_H(F, 9)$ is retract rational over F (Theorem 4.11), via Saltman's retract rationality theorem [16, Corollary 3.13].

Hypotheses. The original statement in [1, Problem 5.2] is made over an arbitrary field. The cohomological computations ($\mathrm{H}^2(H, M|_H) \cong C_9$ and $\exp(\mathrm{H}^2(H, P)) \mid 3$) are purely integral, valid whenever $\mathrm{char}\, F \nmid 9$. The geometric conclusions (non-stable-rationality, non-rationality) require $\mathrm{char}\, F = 0$ via Theorem 2.6, which depends on resolution of singularities.

Retract rationality holds over any field (Theorem 4.11). Neither non-stable-rationality nor non-rationality relies on the open converse Endo–Miyata implication.

Generalizability. The proof generalizes verbatim to all odd primes p (Theorem 5.1 and §5).

Notation. C_n denotes the cyclic group of order n. For the main case (Problem 5.2), $H = C_3 \times C_3 = \langle a, b \mid a^3 = b^3 = 1,\ ab = ba \rangle$; the generalization to $H = C_p \times C_p$ is treated in §5. The base field F is of characteristic 0. For a finite group G, a G-lattice is a finitely generated $\mathbb{Z}[G]$-module that is $\mathbb{Z}$-free. The group ring is denoted $\mathbb{Z}[G]$. The augmentation ideal is $I[G] = \ker(\mathbb{Z}[G] \twoheadrightarrow \mathbb{Z})$, where the augmentation map sends $\sum n_g g \mapsto \sum n_g$. Cohomology $\mathrm{H}^*(G, M)$ denotes ordinary group cohomology; for $\mathbb{Z}[G]$-lattices this coincides with Tate cohomology in positive degrees [18, Def. 6.2.4]. The notation $\exp(A)$ denotes the exponent of a finite abelian group A (the least positive integer annihilating A).

Declaration of Artificial Intelligence Assistance. In the preparation of this work, we used DeepSeek for initial brainstorming of the literature search strategy, drafting a preliminary version, and proofreading. All AI-generated content was critically reviewed, substantially rewritten, and fact-checked by the authors. The authors take full responsibility for scientific integrity of the entire manuscript.

Acknowledgements. We are supported by Xiangtan University project 09KZ-KZ08101 *algebraic invariants and local-global principles*.

2 Preliminaries

Definition 2.1. A $\mathbb{Z}[G]$-*lattice* is a finitely generated $\mathbb{Z}[G]$-module that is $\mathbb{Z}$-free. A $\mathbb{Z}[G]$-lattice M is called:

1. *permutation* if $M \cong \bigoplus_{i=1}^{r} \mathbb{Z}[G/K_i]$ for subgroups $K_i \leq G$, where $\mathbb{Z}[G/K]$ denotes the $\mathbb{Z}[G]$-module with $\mathbb{Z}$-basis the left cosets G/K and G-action by left multiplication;

2. *stably permutation* if there exist permutation lattices P, Q such that $M \oplus P \cong Q$;

These concepts are standard; see [7, §1].

Lemma 2.2 (Character of a permutation module)**.** *Let $K \leq H$ be finite groups and let $\mathbb{Z}[H/K]$ be the permutation $\mathbb{Z}[H]$-lattice (Definition 2.1). The character of the rational representation $\mathbb{Q}[H/K]$ is*

$$\chi_{H/K}(g) = \big|\{xK \in H/K : gxK = xK\}\big| = \begin{cases} [H:K], & g \in K, \\ 0, & g \notin K. \end{cases}$$

Moreover, for an abelian group H, the multiplicities m_K in a permutation lattice $P \cong \bigoplus_{K \leq H} m_K \cdot \mathbb{Z}[H/K]$ are uniquely determined by the character of $P \otimes_{\mathbb{Z}} \mathbb{Q}$ [9, Example 10.4].

This lemma is the key tool in Lemma 3.8 below, where the multiplicities of a permutation lattice with a given rational character are determined uniquely from character evaluations.

Definition 2.3. A $\mathbb{Z}[G]$-lattice F is *flasque* if $\hat{\mathrm{H}}^{-1}(K, F) = 0$ for all subgroups $K \leq G$ (equivalently, $\mathrm{Ext}^1_G(F, P) = 0$ for every permutation lattice P) [7, §1, Lemme 9]. Here $\hat{\mathrm{H}}^{-1}$ denotes Tate cohomology in degree -1; for a finite group G and a $\mathbb{Z}[G]$-lattice M, $\hat{\mathrm{H}}^{-1}(G, M) = \ker(N_G)/I[G]M$, where $N_G = \sum_{g \in G} g$ is the norm element and $I[G] = \ker(\mathbb{Z}[G] \twoheadrightarrow \mathbb{Z})$ is the augmentation ideal.

Definition 2.4. Two $\mathbb{Z}[G]$-lattices M, N are called *similar* (written $M \sim N$) if $M \oplus S_1 \cong N \oplus S_2$ for some permutation lattices S_1, S_2 (Definition 2.1). This is an equivalence relation; denote the similarity class of M by $[M]$. Let

$$C(G)/S(G) = \{[M] : M \text{ a finitely generated } \mathbb{Z}[G]\text{-lattice}\},$$

a commutative monoid under direct sum, with zero element the class of permutation lattices. Its maximal subgroup is

$$D(G)/S(G) = \{[M] \in C(G)/S(G) : M \text{ is stably permutation}\},$$

which is the semigroup $C(\Pi)/S(\Pi)$ of [17, §4.7].

Proposition 2.5 (Flasque resolution)**.** *Every $\mathbb{Z}[G]$-lattice M has a* flasque resolution

$$0 \longrightarrow M \longrightarrow P \longrightarrow F \longrightarrow 0$$

with P permutation (Definition 2.1) and F flasque (Definition 2.3) [7, §1, Lemme 3]. The similarity class $[F] \in D(G)/S(G)$ (Definition 2.4) depends only on M [7, §1, Lemme 5]; we denote it by $\rho(M) = [F]$ and call it the flasque class *of M. By construction, M is stably permutation iff $\rho(M) = 0$; this is the defining property of the group $D(G)/S(G)$.*

Theorem 2.6 (Endo–Miyata, Colliot-Thélène–Sansuc, Voskresenskii)**.** *Let G be a finite group and M a $\mathbb{Z}[G]$-lattice on which G acts faithfully. Let F be a field of characteristic* 0*. Here $F(M)$ denotes the field of rational functions of the torus with character lattice M (the fraction field of the group algebra $F[M]$, a purely transcendental extension of F), on which G acts via the $\mathbb{Z}[G]$-module structure of M; $F(M)^G$ is its G-invariant subfield. Then $F(M)^G$ is* stably rational *over F if and only if M is a stably permutation $\mathbb{Z}[G]$-lattice [17, 4.7, theorem 2] (i.e., its flasque class $\rho(M) \in D(G)/S(G)$ vanishes [7, §1, Lemme 6]).*

In the sequel we apply this theorem in the contrapositive direction: if $M|_H$ is *not* stably permutation, then $F(M|_H)^H \cong Z_H(F, 9)$ is *not* stably rational. The cohomological computation of §4 proves that $\mathrm{H}^2(H, M|_H) \cong C_9$ while permutation lattices have H^2-exponent dividing 3; hence $M|_H$ is not stably permutation (Theorem 4.6), and the theorem yields non-stable-rationality (Corollary 4.7). The same strategy generalizes to $C_p \times C_p$ in §5.

Theorem 2.7 (Universal Coefficient Theorem)**.** *For a finite group G and a trivial G-module M, there is a natural short exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_{n-1}(G, \mathbb{Z}), M) \longrightarrow \mathrm{H}^n(G, M) \longrightarrow \mathrm{Hom}(\mathrm{H}_n(G, \mathbb{Z}), M) \longrightarrow 0$$

for all $n \geq 1$ [18, 3.6.5]. If M is divisible (e.g. $M = \mathbb{Q}/\mathbb{Z}$), the sequence splits, giving a direct sum decomposition.

This theorem is used in Proposition 5.3 to compute $\mathrm{H}^2(H, \mathbb{Q}/\mathbb{Z})$.

3 The Procesi Lattice

We give a self-contained construction of the Procesi $\mathbb{Z}[S_9]$-lattice M and the Procesi isomorphism, keeping only what is needed for the rationality computations. An accessible modern exposition is in [1, §5] and [12]; the original reference is [15].

3.1 Construction over S_9

Let $n = 9$ and let $\Omega = S_9/S_8$ be the set of left cosets of S_8 in S_9. Then $|\Omega| = 9$ and S_9 acts on Ω by left multiplication. Define the permutation $\mathbb{Z}[S_9]$-module

$$U = \mathbb{Z}[\Omega] \cong \bigoplus_{\omega \in \Omega} \mathbb{Z} \cdot e_\omega,$$

where $\{e_\omega\}_{\omega\in\Omega}$ is the standard basis and $g\cdot e_\omega = e_{g\omega}$ for $g\in S_9$. Consider the $\mathbb{Z}[S_9]$-module map

$$\varphi : U\otimes_{\mathbb{Z}} U \longrightarrow \mathbb{Z}[\Omega], \qquad \varphi(e_\alpha\otimes e_\beta) = e_\alpha - e_\beta,$$

extended $\mathbb{Z}$-linearly. The map φ is S_9-equivariant:

$$\varphi\big(g\cdot(e_\alpha\otimes e_\beta)\big) = \varphi(e_{g\alpha}\otimes e_{g\beta}) = e_{g\alpha}-e_{g\beta} = g\cdot(e_\alpha - e_\beta) = g\cdot\varphi(e_\alpha\otimes e_\beta).$$

The image of φ is the *augmentation submodule*

$$I[\Omega] = \ker\Big(\varepsilon : \mathbb{Z}[\Omega]\twoheadrightarrow\mathbb{Z},\ \varepsilon\big(\sum a_\omega e_\omega\big) = \sum a_\omega\Big).$$

Indeed, every difference $e_\alpha - e_\beta$ has augmentation zero, and conversely any element $\sum a_\omega e_\omega$ with $\sum a_\omega = 0$ is a $\mathbb{Z}$-linear combination of such differences (take $\{e_\alpha - e_{\omega_0} : \alpha\neq\omega_0\}$ for a fixed ω_0). Moreover, φ is surjective onto $I[\Omega]$.

Definition 3.1. The *Procesi* $\mathbb{Z}[S_9]$*-lattice* is $M = \ker(\varphi)$.

By construction we have the short exact sequence of $\mathbb{Z}[S_9]$-modules

$$0\longrightarrow M\longrightarrow U\otimes_{\mathbb{Z}} U \xrightarrow{\ \varphi\ } I[\Omega]\longrightarrow 0. \tag{1}$$

The module M is $\mathbb{Z}$-free: it is a $\mathbb{Z}$-submodule of the free $\mathbb{Z}$-module $U\otimes_{\mathbb{Z}}U$, and every submodule of a free $\mathbb{Z}$-module is free (since $\mathbb{Z}$ is a principal ideal domain) [5, Ch.VII, §3, Th.1, Cor.2]. The ranks are

$$\operatorname{rk}_{\mathbb{Z}} U = 9, \qquad \operatorname{rk}_{\mathbb{Z}}(U\otimes U) = 81, \qquad \operatorname{rk}_{\mathbb{Z}} I[\Omega] = 8, \qquad \operatorname{rk}_{\mathbb{Z}} M = 73.$$

Remark 3.2. Procesi [15] proved that the center of the generic matrix algebra satisfies $Z(F,9)\cong F(M)^{S_9}$, where $F(M)^{S_9}$ is the invariant field of the purely transcendental extension $F(M)$ under the natural S_9-action induced by the $\mathbb{Z}[S_9]$-module structure of M. For any subgroup $H\subseteq S_9$, taking H-invariants gives

$$Z_H(F,9) = F(M)^H \cong F(M|_H)^H,$$

where $M|_H$ denotes M viewed as a $\mathbb{Z}[H]$-module by restriction. This isomorphism holds over any field F of characteristic 0; see [1, §5], [12], and the references therein. For the computations that follow, only the exact sequence (1) and this identification are required.

3.2 Restriction to $H = C_3 \times C_3$

Now fix a transitive embedding $H = C_3 \times C_3 \hookrightarrow S_9$, i.e., an embedding whose image acts transitively on $\Omega = S_9/S_8$. Since $|H| = 9 = |\Omega|$, the action is moreover regular (i.e., free and transitive, called *simply transitive* in [4, ch.I, §5, no.6, def.7]): choosing a basepoint $\omega_0 \in \Omega$, the map $H \to \Omega$, $h \mapsto h \cdot \omega_0$, is an H-equivariant bijection. Consequently we obtain an explicit isomorphism of $\mathbb{Z}[H]$-modules

$$U|_H \xrightarrow{\cong} \mathbb{Z}[H], \qquad e_{h\cdot\omega_0} \longmapsto h \quad (h \in H). \tag{2}$$

Proposition 3.3 (Independence of basepoint choice)**.** *The isomorphism* (2) *is independent of the choice of ω_0 up to a $\mathbb{Z}[H]$-module automorphism of $\mathbb{Z}[H]$. Consequently, $\mathrm{Ext}^1_{\mathbb{Z}[H]}(-, M|_H)$, $\mathrm{H}^2(H, M|_H)$, and the isomorphism class of $M|_H$ are independent of this choice.*

Proof. Let $\omega_0' = g \cdot \omega_0$ be another basepoint. The isomorphism induced by ω_0' sends $e_{h\cdot\omega_0'} = e_{hg\cdot\omega_0} \longmapsto h$, while the original sends $e_{hg\cdot\omega_0} \longmapsto hg$; hence the two isomorphisms differ by right multiplication by g^{-1} on $\mathbb{Z}[H]$. Right multiplication by a fixed element of H is a $\mathbb{Z}[H]$-module automorphism of the left regular module. Therefore the induced map on cohomology and Ext^1 is the identity, and the isomorphism class of $M|_H = \ker(\pi|_H)$ is unchanged. □

Proposition 3.4 (Independence of the embedding)**.** *Let $\iota_1, \iota_2 : H \hookrightarrow S_9$ be two transitive embeddings. Then $M|_{\iota_1(H)} \cong M|_{\iota_2(H)}$ as $\mathbb{Z}[H]$-modules.*

Proof. Both ι_1 and ι_2 make Ω into a regular H-set (since $|H| = 9 = |\Omega|$ and each action is transitive). Any two regular H-sets of the same cardinality are H-equivariantly isomorphic: fix basepoints $\omega_1, \omega_2 \in \Omega$, and define a bijection $\Omega \to \Omega$ by $h \cdot \omega_1 \longmapsto h \cdot \omega_2$. This bijection is an element $\sigma \in S_9$, and by construction $\sigma\, \iota_1(h)\, \sigma^{-1} = \iota_2(h)$ for all $h \in H$. Conjugating the embedding by σ replaces the H-action on $U \otimes U$ with its conjugate under the S_9-action, which induces an isomorphism $M|_{\iota_1(H)} \cong M|_{\iota_2(H)}$ of $\mathbb{Z}[H]$-modules via the identification of H with $\iota_1(H)$ and $\iota_2(H)$ followed by the conjugation $\iota_2^{-1} \circ \iota_1$. □

Proposition 3.5. *There is an isomorphism of $\mathbb{Z}[H]$-modules*

$$\mathbb{Z}[H] \otimes_{\mathbb{Z}} \mathbb{Z}[H] \;\cong\; \mathbb{Z}[H]^{\oplus 9},$$

where the left side carries the diagonal action $h \cdot (x \otimes y) = hx \otimes hy$ and the right side carries componentwise left multiplication.

Proof. Identify $\mathbb{Z}[H]$ with the induced module $\mathrm{Ind}_1^H(\mathbb{Z}) = \mathbb{Z}[H] \otimes_{\mathbb{Z}} \mathbb{Z}$, where Ind_1^H denotes induction from the trivial subgroup $\{1\}$ to H, and Res_1^H denotes restriction from H to $\{1\}$. For $\mathbb{Z}$-modules A, B with trivial H-action, the projection formula for induced modules [9, Corollary 10.20] states

$$\mathrm{Ind}_1^H(A) \otimes_{\mathbb{Z}} \mathrm{Ind}_1^H(B) \;\cong\; \mathrm{Ind}_1^H\big(A \otimes_{\mathbb{Z}} \mathrm{Res}_1^H \mathrm{Ind}_1^H(B)\big)$$

as $\mathbb{Z}[H]$-modules (diagonal action on the left, induced action on the right). Taking $A = B = \mathbb{Z}$ gives

$$\mathbb{Z}[H] \otimes_{\mathbb{Z}} \mathbb{Z}[H] \;\cong\; \mathrm{Ind}_1^H\big(\mathrm{Res}_1^H \mathrm{Ind}_1^H(\mathbb{Z})\big).$$

The restriction $\mathrm{Res}_1^H \mathrm{Ind}_1^H(\mathbb{Z})$ is the regular module $\mathbb{Z}[H]$ viewed merely as a $\mathbb{Z}$-module (forgetting the H-action); by definition $\mathbb{Z}[H]$ has $\mathbb{Z}$-basis $\{h : h \in H\}$, so $\mathrm{Res}_1^H \mathrm{Ind}_1^H(\mathbb{Z}) \cong \mathbb{Z}^{\oplus 9}$. Applying Ind_1^H then yields $\mathrm{Ind}_1^H(\mathbb{Z}^{\oplus 9}) \cong \mathrm{Ind}_1^H(\mathbb{Z})^{\oplus 9} \cong \mathbb{Z}[H]^{\oplus 9}$. The isomorphism is H-equivariant: choose the $\mathbb{Z}$-basis $\{g\}_{g \in H}$ of the second tensor factor $\mathbb{Z}[H]$; the map $x \otimes g \longmapsto x$ in the g-th coordinate satisfies $h \cdot (x \otimes g) = hx \otimes hg \longmapsto hx$ in the hg-th coordinate. □

Under the identification (2) and Proposition 3.5, the restriction of the map φ to H yields the following.

Proposition 3.6 (Procesi sequence for H)**.** *There is an exact sequence of $\mathbb{Z}[H]$-modules*

$$0 \longrightarrow M|_H \longrightarrow \mathbb{Z}[H]^{\oplus 9} \xrightarrow{\pi|_H} I[H] \longrightarrow 0, \tag{3}$$

where $I[H] = \ker(\varepsilon : \mathbb{Z}[H] \twoheadrightarrow \mathbb{Z})$, $\varepsilon(\sum a_h h) = \sum a_h$, is the augmentation ideal of H. All modules in the sequence are finitely generated and $\mathbb{Z}$-free, and $\mathrm{rk}_{\mathbb{Z}} M|_H = 73$.

Proof. Exactness. Under the identifications, the map φ becomes a $\mathbb{Z}[H]$-module map $\pi|_H : \mathbb{Z}[H]^{\oplus 9} \longrightarrow I[H]$. Each standard basis element of $\mathbb{Z}[H]^{\oplus 9}$ maps to a difference $h - h'$ in $\mathbb{Z}[H]$; these differences generate the $\mathbb{Z}$-module $I[H]$, and $I[H]$ is generated as a $\mathbb{Z}[H]$-module by any single element $h - 1$ with $h \neq 1$; hence $\pi|_H$ is surjective. Define $M|_H = \ker(\pi|_H)$, the restriction of the Procesi lattice to H.

Freeness. All modules in the sequence are finitely generated and $\mathbb{Z}$-free: $\mathbb{Z}[H]^{\oplus 9}$ is $\mathbb{Z}$-free with basis the group elements in each coordinate; $I[H]$ is a $\mathbb{Z}$-submodule of $\mathbb{Z}[H]$, hence free [5, Ch.VII, §3, Th.1, Cor.2]; $M|_H = \ker(\pi|_H)$ is a $\mathbb{Z}$-submodule of $\mathbb{Z}[H]^{\oplus 9}$, hence free.

Rank. $\mathrm{rk}_{\mathbb{Z}} M|_H = \mathrm{rk}_{\mathbb{Z}} \mathbb{Z}[H]^{\oplus 9} - \mathrm{rk}_{\mathbb{Z}} I[H] = 9 \cdot 9 - 8 = 73$, matching the rank of M over S_9. □

Proposition 3.7 (Rational character of $M|_H$)**.**

$$M|_H \otimes_{\mathbb{Z}} \mathbb{Q} \;\cong\; 8 \cdot \mathbb{Q}[H] \;\oplus\; \mathbb{Q},$$

where $\mathbb{Q}$ *denotes the one-dimensional trivial* $\mathbb{Q}[H]$*-module.*

Proof. Tensor the exact sequence of Proposition 3.6 with $\mathbb{Q}$ (flat over $\mathbb{Z}$, [3, Ch.II, §2, no.4, Th.1]):

$$0 \longrightarrow M|_H \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}[H]^{\oplus 9} \longrightarrow I[H] \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow 0.$$

The augmentation sequence

$$0 \longrightarrow I[H] \longrightarrow \mathbb{Z}[H] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{4}$$

remains exact after tensoring with $\mathbb{Q}$ (flat over $\mathbb{Z}$, [3, Ch.II, §2, no.4, Th.1]), giving

$$0 \longrightarrow I[H] \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}[H] \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0.$$

This sequence splits since $|H| = 9$ is invertible in $\mathbb{Q}$ (Maschke's Theorem [9, (3.14)]; the splitting is given by the idempotent $e_0 = \frac{1}{|H|}\sum_{h \in H} h$). Hence in the Grothendieck group $K_0(\mathbb{Q}[H])$ [9, (16.3)],

$$[I[H] \otimes_{\mathbb{Z}} \mathbb{Q}] = [\mathbb{Q}[H]] - [\mathbb{Q}].$$

From the exact sequence of $\mathbb{Q}[H]$-modules above,

$$\begin{aligned} [M|_H \otimes \mathbb{Q}] &= 9[\mathbb{Q}[H]] - [I[H] \otimes \mathbb{Q}] \\ &= 9[\mathbb{Q}[H]] - ([\mathbb{Q}[H]] - [\mathbb{Q}]) \\ &= 8[\mathbb{Q}[H]] + [\mathbb{Q}]. \end{aligned}$$

Since $\mathbb{Q}[H]$ is semisimple by Maschke [9, (3.14)], equality in $K_0(\mathbb{Q}[H])$ implies

$$M|_H \otimes_{\mathbb{Z}} \mathbb{Q} \cong 8 \cdot \mathbb{Q}[H] \oplus \mathbb{Q}$$

of $\mathbb{Q}[H]$-modules. □

From Proposition 3.7 we deduce: if $M|_H$ *were* a permutation $\mathbb{Z}[H]$-lattice, its rational character would force $M|_H \cong 8 \cdot \mathbb{Z}[H] \oplus \mathbb{Z}$.

Lemma 3.8. *Let* P *be a permutation* $\mathbb{Z}[H]$*-lattice satisfying* $P \otimes_{\mathbb{Z}} \mathbb{Q} \cong 8 \cdot \mathbb{Q}[H] \oplus \mathbb{Q}$*. Then* $P \cong 8 \cdot \mathbb{Z}[H] \oplus \mathbb{Z}$*.*

Proof. Since P is a permutation $\mathbb{Z}[H]$-lattice (Definition 2.1), we can write

$$P \cong \bigoplus_{K \leq H} m_K \cdot \mathbb{Z}[H/K], \qquad m_K \in \mathbb{Z}_{\geq 0},$$

where K runs over the subgroups of $H = C_3 \times C_3$. The subgroups are: $\{1\}$ (trivial, index 9); four subgroups of order 3 (each index 3), namely $K_1 = \langle a \rangle$, $K_2 = \langle b \rangle$, $K_3 = \langle ab \rangle$, $K_4 = \langle ab^2 \rangle$ where $H = \langle a \rangle \times \langle b \rangle \cong C_3 \times C_3$; and H itself (index 1, giving the trivial module $\mathbb{Z}$).

Character-theoretic determination of multiplicities. By Lemma 2.2, the character of $\mathbb{Q}[H/K]$ is given by

$$\chi_{H/K}(g) = \begin{cases} [H : K], & g \in K, \\ 0, & g \notin K, \end{cases}$$

and the multiplicities m_K are uniquely determined by the character of P.

By Proposition 3.7, $\chi_P = 8\chi_{H/\{1\}} + \chi_{H/H}$, so $\chi_P(1) = 8 \cdot 9 + 1 = 73$ and $\chi_P(g) = 0 + 1 = 1$ for all $g \neq 1$.

System of equations. Let $a = m_{\{1\}}$, $b_j = m_{K_j}$ $(j = 1, \ldots, 4)$, and $c = m_H$. Evaluating χ_P at the identity and at $g \in K_j \setminus \{1\}$ gives

$$\chi_P(1) = 9a + 3(b_1 + b_2 + b_3 + b_4) + c = 73, \tag{5}$$

$$\chi_P(g) = 3b_j + c = 1 \qquad (g \in K_j,\ g \neq 1,\ j = 1, \ldots, 4). \tag{6}$$

Solution. Equation (6) gives $b_j = (1-c)/3$. Since the multiplicities b_j are non-negative integers by definition of a permutation lattice decomposition, we must have $c = 1$, forcing $b_j = 0$ for all j. Substituting into (5): $9a+0+1 = 73$, hence $a = 8$. Therefore $P \cong 8 \cdot \mathbb{Z}[H] \oplus \mathbb{Z}$. □

4 Cohomological Obstruction to Stable Permutation

In this section we compute the second integral cohomology group $\mathrm{H}^2(H, M|_H)$ and prove that its exponent obstructs stable permutation.

Notation. Throughout this section we write $\mathrm{H}^q(G, M)$ for the q-th integral group cohomology of a G-module M. The upright H is reserved for the cohomology functor, while the italic H denotes the group $H = C_3 \times C_3$.

4.1 Cohomology Computations

Lemma 4.1. *The group* $\mathrm{H}^1(H, I[H])$ *is cyclic of order* 9.

Proof. Apply group cohomology $\mathrm{H}^*(H, -)$ to the augmentation exact sequence (4). The associated long exact cohomology sequence reads

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(H, I[H]) \longrightarrow \mathrm{H}^0(H, \mathbb{Z}[H]) \longrightarrow \mathrm{H}^0(H, \mathbb{Z}) \\ \longrightarrow \mathrm{H}^1(H, I[H]) \longrightarrow \mathrm{H}^1(H, \mathbb{Z}[H]) \longrightarrow \cdots . \end{aligned}$$

We analyse the low-degree terms:

- $\mathrm{H}^0(H, \mathbb{Z}[H]) = \mathbb{Z}[H]^H = \mathbb{Z} \cdot N_H$, where $N_H = \sum_{h \in H} h$ is the norm element.
- $\mathrm{H}^0(H, \mathbb{Z}) = \mathbb{Z}^H = \mathbb{Z}$.
- The map $\mathrm{H}^0(H, \mathbb{Z}[H]) \to \mathrm{H}^0(H, \mathbb{Z})$ sends $N_H \mapsto \varepsilon(N_H) = \sum_{h \in H} 1 = |H| = 9$.
- $\mathrm{H}^1(H, \mathbb{Z}[H]) = 0$: $\mathbb{Z}[H] \cong \mathrm{Ind}_1^H \mathbb{Z}$ as $\mathbb{Z}[H]$-modules, and Shapiro's Lemma [18, Theorem 6.3.2] gives $\mathrm{H}^1(H, \mathrm{Ind}_1^H \mathbb{Z}) \cong \mathrm{H}^1(\{1\}, \mathbb{Z}) = 0$.

From the exactness at $\mathrm{H}^0(H, \mathbb{Z})$ we obtain $\operatorname{coker}(\mathbb{Z} \cdot N_H \xrightarrow{\varepsilon} \mathbb{Z}) = \mathbb{Z}/9\mathbb{Z}$, and exactness at $\mathrm{H}^1(H, I[H])$ with the vanishing of $\mathrm{H}^1(H, \mathbb{Z}[H])$ yields the short exact sequence

$$0 \longrightarrow \mathbb{Z}/9\mathbb{Z} \longrightarrow \mathrm{H}^1(H, I[H]) \longrightarrow 0,$$

hence $\mathrm{H}^1(H, I[H]) \cong \mathbb{Z}/9\mathbb{Z} \cong C_9$. □

Lemma 4.2. *For the Procesi restriction* $M|_H$, $\mathrm{H}^2(H, M|_H) \cong C_9$.

Proof. Apply $\mathrm{H}^*(H, -)$ to the exact sequence of Proposition 3.6:

$$0 \longrightarrow M|_H \longrightarrow \mathbb{Z}[H]^{\oplus 9} \longrightarrow I[H] \longrightarrow 0.$$

The associated long exact cohomology sequence gives

$$\begin{aligned} \cdots \longrightarrow \mathrm{H}^1(H, \mathbb{Z}[H]^{\oplus 9}) \longrightarrow \mathrm{H}^1(H, I[H]) \\ \longrightarrow \mathrm{H}^2(H, M|_H) \longrightarrow \mathrm{H}^2(H, \mathbb{Z}[H]^{\oplus 9}) \longrightarrow \cdots . \end{aligned}$$

By Shapiro's Lemma [18, Theorem 6.3.2], $\mathrm{H}^i(H, \mathbb{Z}[H]) \cong \mathrm{H}^i(\{1\}, \mathbb{Z}) = 0$ for all $i > 0$. Hence $\mathrm{H}^1(H, \mathbb{Z}[H]^{\oplus 9}) = \mathrm{H}^2(H, \mathbb{Z}[H]^{\oplus 9}) = 0$, and the long exact sequence collapses to the isomorphism

$$\mathrm{H}^2(H, M|_H) \;\cong\; \mathrm{H}^1(H, I[H]).$$

By Lemma 4.1, $\mathrm{H}^1(H, I[H]) \cong C_9$. □

Lemma 4.3. *With the trivial H-action on $\mathbb{Z}$, $\mathrm{H}^2(H,\mathbb{Z}) \cong C_3 \times C_3$.*

Proof. Apply $\mathrm{H}^*(H,-)$ to the short exact sequence of trivial H-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

The associated long exact sequence reads

$$\begin{aligned} 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} & \\ \longrightarrow \mathrm{H}^1(H,\mathbb{Z}) \longrightarrow \mathrm{H}^1(H,\mathbb{Q}) \longrightarrow \mathrm{H}^1(H,\mathbb{Q}/\mathbb{Z}) & \\ \longrightarrow \mathrm{H}^2(H,\mathbb{Z}) \longrightarrow \mathrm{H}^2(H,\mathbb{Q}) \longrightarrow \cdots . & \end{aligned}$$

Since $\mathbb{Q}$ is uniquely divisible, multiplication by 3 is an automorphism of $\mathbb{Q}$; hence $\mathrm{H}^i(H,\mathbb{Q}) = 0$ for all $i > 0$ [14, Proposition 1.6.2(c)]. Substituting $\mathrm{H}^1(H,\mathbb{Q}) = \mathrm{H}^2(H,\mathbb{Q}) = 0$ yields $\mathrm{H}^2(H,\mathbb{Z}) \cong \mathrm{H}^1(H,\mathbb{Q}/\mathbb{Z})$. Since $\mathbb{Q}/\mathbb{Z}$ carries the trivial H-action, $\mathrm{H}^1(H,\mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(H,\mathbb{Q}/\mathbb{Z})$ [18, Corollary 6.4.6]. Now $\mathrm{Hom}(H,\mathbb{Q}/\mathbb{Z}) \cong C_3 \times C_3$ because $\mathrm{Hom}(C_3,\mathbb{Q}/\mathbb{Z}) \cong C_3$ ($\mathbb{Q}/\mathbb{Z}$ has a unique subgroup of order 3) and $\mathrm{Hom}(-,\mathbb{Q}/\mathbb{Z})$ preserves finite direct sums [4, Ch.II, §1, no.6, Prop.6, Cor.1]. Hence $\mathrm{H}^2(H,\mathbb{Z}) \cong C_3 \times C_3$. □

4.2 Exponent of H^2 for Permutation Lattices

Proposition 4.4. *Let P be a permutation $\mathbb{Z}[H]$-lattice. Then every element of $\mathrm{H}^2(H,P)$ has order dividing 3; equivalently, $\exp(\mathrm{H}^2(H,P)) \mid 3$.*

Proof. Write $P \cong \bigoplus_i \mathbb{Z}[H/K_i]$ for subgroups $K_i \leq H$. Since $\mathrm{H}^2(H,-)$ is additive and therefore preserves finite direct sums [18, Proposition 3.3.4], $\mathrm{H}^2(H,P) \cong \bigoplus_i \mathrm{H}^2(H,\mathbb{Z}[H/K_i])$. By Shapiro's Lemma [18, Theorem 6.3.2], $\mathbb{Z}[H/K_i] \cong \mathrm{Ind}_{K_i}^H(\mathbb{Z})$, hence $\mathrm{H}^2(H,\mathbb{Z}[H/K_i]) \cong \mathrm{H}^2(K_i,\mathbb{Z})$, where $\mathbb{Z}$ carries the trivial K_i-action. For each subgroup $K \leq H$ we compute $\mathrm{H}^2(K,\mathbb{Z})$:

- $K = \{1\}$: $\mathrm{H}^2(\{1\},\mathbb{Z}) = 0$.
- $K \cong C_3$ (four subgroups of order 3): $\mathrm{H}^2(C_3,\mathbb{Z}) \cong \mathrm{Hom}(C_3,\mathbb{Q}/\mathbb{Z}) \cong C_3$.
- $K = H = C_3 \times C_3$: $\mathrm{H}^2(H,\mathbb{Z}) \cong C_3 \times C_3$ (Lemma 4.3).

In every case $\exp(\mathrm{H}^2(K,\mathbb{Z})) \mid 3$. Hence $\exp(\mathrm{H}^2(H,P)) \mid 3$. □

Proposition 4.5. *If a $\mathbb{Z}[H]$-lattice L is stably permutation, then*

$$\exp(\mathrm{H}^2(H,L)) \mid 3.$$

Proof. Write $L \oplus P \cong Q$ with P, Q permutation. By additivity [18, Proposition 3.3.4], $\mathrm{H}^2(H,L) \oplus \mathrm{H}^2(H,P) \cong \mathrm{H}^2(H,Q)$. Both $\mathrm{H}^2(H,P)$ and $\mathrm{H}^2(H,Q)$ have exponent dividing 3 by Proposition 4.4. For any $x \in \mathrm{H}^2(H,L)$, the element $(x,0) \in \mathrm{H}^2(H,Q)$ satisfies $3(x,0) = 0$; hence $3x = 0$. Therefore $\exp(\mathrm{H}^2(H,L)) \mid 3$. □

4.3 The Obstruction

Theorem 4.6 (Non-stably-permutation)**.** *The restricted Procesi lattice $M|_H$ is not a stably permutation $\mathbb{Z}[H]$-lattice.*

Proof. By Lemma 4.2, $\mathrm{H}^2(H, M|_H) \cong C_9$, a group of exponent 9. If $M|_H$ were stably permutation, Proposition 4.5 would force $\exp(\mathrm{H}^2(H, M|_H)) \mid 3$. Contradiction. □

Corollary 4.7 (Non-stably-rational)**.** *The field $Z_H(F, 9)$ is not stably rational over F.*

Proof. Recall that $Z_H(F, 9) \cong F(M|_H)^H$ (Remark 3.2). By Theorem 4.6, $M|_H$ is not stably permutation. By Theorem 2.6, $F(M|_H)^H$ is stably rational over F if and only if $M|_H$ is stably permutation. Hence $Z_H(F, 9)$ is not stably rational. □

Remark 4.8. The purely lattice-theoretic part (Theorem 4.6) holds integrally, hence for any field whose characteristic does not divide $|H| = 9$. The geometric conclusion (Theorem 4.7) additionally requires characteristic 0 via Theorem 2.6.

Proposition 4.9 (Faithfulness of the H-action)**.** *Let $H = C_3 \times C_3$ act on $M_9(F)$ via the regular representation $H \hookrightarrow S_9$ by permutation of matrix indices: $h \cdot e_{ij} = e_{h(i),h(j)}$ for the standard matrix units. The action of H on $M|_H$ is the restriction of the diagonal left-regular action on $\mathbb{Z}[H]^{\oplus 9}$:*

$$h \cdot (x_1, \ldots, x_9) = (hx_1, \ldots, hx_9), \qquad h \in H,\ (x_1, \ldots, x_9) \in M|_H \subset \mathbb{Z}[H]^{\oplus 9}.$$

This action is faithful.

Proof. The action on $M_9(F)$ is faithful because $h \neq 1$ implies $h(i) \neq i$ for some index i, whence $h \cdot e_{ij} \neq e_{ij}$. The Procesi isomorphism $Z_H(F, 9) \cong F(M|_H)^H$ (Remark 3.2) intertwines the H-actions. Any $h \in H$ acting trivially on $M|_H$ would, by functoriality of the field of fractions, act trivially on $F(M|_H) = \mathrm{Frac}(\mathrm{Sym}_F(M|_H \otimes_{\mathbb{Z}} F))$ and therefore on $F(M|_H)^H \cong Z_H(F, 9)$, hence on $M_9(F)$ — contradicting faithfulness on $M_9(F)$. □

Theorem 4.10 (Saltman, 1984)**.** *[16, Corollary 3.13]. Let A be a finite abelian group, and let 2^r be the highest power of 2 dividing the exponent of A. Assume F is a field such that either F has characteristic 2 or $F(\rho)/F$ is a cyclic extension, where ρ is a primitive 2^r-th root of unity. Let V be a finitely generated $F[A]$-module on which A acts faithfully. Then $F(V)^A$ is retract rational over F.*

Corollary 4.11 (Retract rationality)**.** *The field $Z_H(F,9)$ is retract rational over F (Remark 3.2).*

Proof. $Z_H(F,9) \cong F(M|_H)^H$ (Remark 3.2). Let $V = M|_H \otimes_{\mathbb{Z}} F$; then $F(V)^H \cong F(M|_H)^H$. Apply Theorem 4.10 with $A = H = C_3 \times C_3$. The exponent of H is 3 (odd), so $2^r = 1$ and $\rho = 1$; the condition on $F(1)/F$ is vacuous. The group H acts faithfully on V because it acts faithfully on $M|_H$ (Proposition 4.9) and $\operatorname{char} F = 0$. Hence $Z_H(F,9)$ is retract rational over F. □

Proposition 4.12. *The restricted Procesi lattice $M|_H$ is not a permutation $\mathbb{Z}[H]$-lattice.*

Proof. Apply $\mathrm{Hom}_{\mathbb{Z}[H]}(-, L)$ to (4). The associated long exact sequence reads

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}, L) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[H], L) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[H]}(I[H], L) \\ &\longrightarrow \mathrm{Ext}^1_{\mathbb{Z}[H]}(\mathbb{Z}, L) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}[H]}(\mathbb{Z}[H], L) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}[H]}(I[H], L) \\ &\longrightarrow \mathrm{Ext}^2_{\mathbb{Z}[H]}(\mathbb{Z}, L) \longrightarrow \mathrm{Ext}^2_{\mathbb{Z}[H]}(\mathbb{Z}[H], L) \longrightarrow \cdots . \end{aligned}$$

Since $\mathbb{Z}[H]$ is projective, $\mathrm{Ext}^i_{\mathbb{Z}[H]}(\mathbb{Z}[H], -) = 0$ for all $i \geq 1$. Substituting this collapses the sequence to an isomorphism $\mathrm{Ext}^1_{\mathbb{Z}[H]}(I[H], L) \cong \mathrm{Ext}^2_{\mathbb{Z}[H]}(\mathbb{Z}, L) = \mathrm{H}^2(H, L)$. Taking $L = M|_H$ and $L = 8 \cdot \mathbb{Z}[H] \oplus \mathbb{Z}$ yields

$$\mathrm{Ext}^1_{\mathbb{Z}[H]}(I[H], M|_H) \cong C_9, \qquad \mathrm{Ext}^1_{\mathbb{Z}[H]}(I[H], 8 \cdot \mathbb{Z}[H] \oplus \mathbb{Z}) \cong C_3 \times C_3.$$

If $M|_H$ were isomorphic to $8 \cdot \mathbb{Z}[H] \oplus \mathbb{Z}$, these groups would coincide, contradicting $C_9 \ncong C_3 \times C_3$. □

Remark 4.13. Proposition 4.12 is strictly weaker than Theorem 4.6 (which proves non-*stable* permutation). The gap is significant: if M is a permutation lattice, then $F(M)^G$ is stably rational [10, Theorem 1.2]. The converse "$F(M)^G$ rational $\Rightarrow$ M stably permutation" is known for groups with cyclic Sylow subgroups [11, Theorem 1.5], but is *open* for groups with non-cyclic Sylow subgroups (including $H = C_3 \times C_3$). Therefore ruling out $M|_H$ being a permutation lattice does *not* imply non-rationality; that would require the open converse, whereas our main proof uses only the proved equivalence "stably rational $\Leftrightarrow$ stably permutation" (Theorem 2.6).

5 Generalization and comparison

The H^2 obstruction introduced in this paper is a purely integral cohomological invariant of the $\mathbb{Z}[H]$-lattice $M|_H$, distinct from classical geometric invariants

such as the unramified Brauer group and Swan–Endo–Miyata modules. In this section we situate our obstruction among these invariants and discuss its scope.

5.1 Generalization to $C_p \times C_p$

Theorem 5.1 (Obstruction for $C_p \times C_p$)**.** *Let p be an odd prime and let $H = C_p \times C_p \subset S_{p^2}$ act regularly (i.e., freely and transitively on $\{1, 2, \ldots, p^2\}$). Let $M|_H$ be the restriction of the Procesi $\mathbb{Z}[S_{p^2}]$-lattice to H. By the regular identification $U|_H \cong \mathbb{Z}[H]$ and the projection formula $\mathbb{Z}[H] \otimes_{\mathbb{Z}} \mathbb{Z}[H] \cong \mathbb{Z}[H]^{\oplus p^2}$ (Proposition 3.5), $M|_H$ fits into the exact sequence of $\mathbb{Z}[H]$-modules*

$$0 \longrightarrow M|_H \longrightarrow \mathbb{Z}[H]^{\oplus p^2} \longrightarrow I[H] \longrightarrow 0,$$

where $I[H] = \ker(\varepsilon\colon \mathbb{Z}[H] \twoheadrightarrow \mathbb{Z})$ is the augmentation ideal and the surjection sends each standard basis element of $\mathbb{Z}[H]^{\oplus p^2}$ to a difference $h - h'$ in $\mathbb{Z}[H]$. Let F be a field of characteristic 0, and let $Z_H(F, p^2)$ denote the center of the generic division algebra of degree p^2 over F (cf. Remark 3.2), with Procesi isomorphism $Z_H(F, p^2) \cong F(M|_H)^H$. Then:

1. *$\mathrm{H}^2(H, M|_H) \cong C_{p^2}$.*
2. *For any permutation $\mathbb{Z}[H]$-lattice P, $\exp(\mathrm{H}^2(H, P)) \mid p$.*
3. *Consequently, $M|_H$ is not stably permutation, and $Z_H(F, p^2)$ is not stably rational (hence not rational) over F.*
4. *$Z_H(F, p^2)$ is retract rational over F.*

Proof. The Procesi restriction to H follows the same construction as in §3: the regular identification $U|_H \cong \mathbb{Z}[H]$ (Proposition 3.3) and the isomorphism $\mathbb{Z}[H] \otimes \mathbb{Z}[H] \cong \mathbb{Z}[H]^{\oplus p^2}$ (Proposition 3.5, whose proof works verbatim with 9 replaced by p^2) hold integrally for all p. The map $\pi|_H : \mathbb{Z}[H]^{\oplus p^2} \to I[H]$ is surjective because $I[H]$ is generated by $h - 1$ ($h \neq 1$), and $M|_H = \ker(\pi|_H)$ is $\mathbb{Z}$-free (submodule of a free $\mathbb{Z}$-module) of $\mathbb{Z}$-rank $p^4 - p^2 + 1$.

(1) The same cohomological computation as in Lemmas 4.1 and 4.2, with 9 replaced by p^2 and $H = C_p \times C_p$, gives $\mathrm{H}^2(H, M|_H) \cong C_{p^2}$.

(2) The same argument as in Proposition 4.4, with the subgroups of $C_p \times C_p$ in place of those of $C_3 \times C_3$, gives $\exp(\mathrm{H}^2(H, P)) \mid p$.

(3) The same exponent argument as in Theorem 4.6 and Corollary 4.7, with p in place of 3, shows that $M|_H$ is not stably permutation and $Z_H(F, p^2)$ is not stably rational.

(4) The same argument as in Theorem 4.11 applies: H is abelian of odd exponent, so Saltman's [16, Corollary 3.13] gives retract rationality. □

Remark 5.2. The proof of Theorem 5.1 holds for all odd primes p without any dependence on $p = 3$ specifically: the regular embedding $H \hookrightarrow S_{p^2}$ (independent of the choice up to conjugation; Proposition 3.4), Proposition 3.5, the cohomological vanishing (Shapiro), and the subgroup analysis of $C_p \times C_p$ are all purely formal and independent of p. Appendix 6.2 provides computational verification for $p = 3$ and $p = 5$ via direct integer Smith normal form on the bar resolution cochain matrices. The converse Endo–Miyata implication (rational $\Rightarrow$ stably permutation for non-cyclic p-groups) remains open but is not required for any of the above conclusions.

We summarize our results on [1, Problem 5.2]:

Property	*Status*	*Method*
Rational	**No (char 0)**	$\mathrm{H}^2(H, M\|_H) \cong C_9$; rational $\Rightarrow$ stably rational
Stably rational	**No (char 0)**	$\mathrm{H}^2(H, M\|_H) \cong C_9$
Retract rational	**Yes (any field)**	Saltman [16, Corollary 3.13]

5.2 Scope and further directions

Non-regular embeddings. Our proof uses the regular embedding $H \hookrightarrow S_{p^2}$ (Proposition 3.4) in three essential ways. We itemize them to clarify what would change for other embeddings.

1. *Identification $U|_H \cong \mathbb{Z}[H]$.* This relies on the action of H on $\Omega = S_{p^2}/S_{p^2-1}$ being regular (free and transitive, hence H-equivariantly isomorphic to the left regular action), and is independent of the basepoint up to a $\mathbb{Z}[H]$-module automorphism (Proposition 3.3). If H acts on Ω with multiple orbits or with non-trivial stabilizers, then $U|_H \cong \bigoplus_{K \leq H} m_K \cdot \mathbb{Z}[H/K]$ where the multiplicities m_K depend on the orbit structure. By Shapiro's lemma [18, Theorem 6.3.2], $\mathbb{Z}[H/K] \cong \mathrm{Ind}_K^H \mathbb{Z}$; the tensor product $\mathrm{Ind}_K^H \mathbb{Z} \otimes \mathrm{Ind}_L^H \mathbb{Z}$ decomposes via Mackey's formula [9, (10.18)] as a direct sum of $\mathrm{Ind}_{K \cap gLg^{-1}}^H \mathbb{Z}$ over double cosets. Since H is abelian, $gLg^{-1} = L$ and every intersection is $K \cap L$. Applying Shapiro's lemma again yields

$$\mathrm{H}^i(H, U|_H \otimes U|_H) \cong \bigoplus_{K,L} m_K m_L \ \mathrm{H}^i(K \cap L, \mathbb{Z}),$$

generally non-zero for $i > 0$, so the connecting-homomorphism isomorphism $\mathrm{H}^2(H, M|_H) \cong \mathrm{H}^1(H, I[H])$ would no longer be an isomorphism, so $\mathrm{H}^2(H, M|_H) \cong C_9$ (Lemma 4.2) would fail.

2. *Breakdown of* $\mathbb{Z}[H] \otimes \mathbb{Z}[H] \cong \mathbb{Z}[H]^{\oplus p^2}$. In the regular case $\mathbb{Z}[H] \otimes_{\mathbb{Z}} \mathbb{Z}[H] \cong \mathbb{Z}[H]^{\oplus p^2}$ (Proposition 3.5), which relies on $\mathrm{Res}_1^H \mathrm{Ind}_1^H \mathbb{Z} \cong \mathbb{Z}^{\oplus p^2}$. For a non-regular $\mathbb{Z}[H/K]$, however, the tensor product $\mathbb{Z}[H/K] \otimes \mathbb{Z}[H/L]$ decomposes via Mackey's formula [9, (10.18)] into summands indexed by double cosets; the simple rank p^2 is replaced by a sum of $[H : K \cap L]$ terms, so the cohomological analysis changes.

3. *Surjectivity* $\mathbb{Z}[H]^{\oplus p^2} \to I[H]$. This is the rightmost map in the Procesi sequence (Proposition 3.6), where $I[H]$ is generated as a $\mathbb{Z}[H]$-module by $h-1$ and the surjection from p^2 copies of $\mathbb{Z}[H]$ is guaranteed by the regular action. For a non-regular permutation lattice, the image of the Procesi map φ is a submodule of $I[\Omega]$ that may be strictly smaller; the exact sequence of Proposition 3.6 would no longer terminate in $I[H]$, requiring analysis of the specific orbit decomposition.

In summary, for a non-regular embedding the short exact sequence

$$0 \longrightarrow M|_H \longrightarrow P_1 \longrightarrow I[\Omega] \longrightarrow 0$$

(with P_1 a permutation lattice not necessarily of the form $\mathbb{Z}[H]^{\oplus n}$) remains exact, but the cohomological analysis of P_1 becomes substantially more involved and the simple exponent obstruction may no longer yield a definitive answer. Computing $\mathrm{H}^2(H, M|_H)$ for one or two non-regular embeddings of $C_3 \times C_3$ into S_n (e.g. the intransitive embedding $C_3 \times C_3 \subset S_3 \times S_3 \subset S_6$) is an **open** problem.

Non-abelian p-groups. For non-abelian groups the situation is more complex. The subgroup cohomology classification used in Proposition 4.4 extends: if G is any finite group and $P = \bigoplus_i \mathbb{Z}[G/K_i]$ is a permutation lattice, then $\exp(\mathrm{H}^2(G, P)) \mid \max_{K \leq G} \exp(\mathrm{H}^2(K, \mathbb{Z}))$ by Shapiro's lemma. The obstruction therefore applies whenever one can find a natural G-lattice M with $\exp(\mathrm{H}^2(G, M)) > \max_{K \leq G} \exp(\mathrm{H}^2(K, \mathbb{Z}))$.

For the Heisenberg group $G = \langle x, y \mid x^p = y^p = [x, y]^p = 1 \rangle$ of order p^3: $G^{\mathrm{ab}} \cong C_p \times C_p$ and $\mathrm{H}_2(G, \mathbb{Z}) \cong C_p \times C_p$ [**?**, Ch. IV, Theorem 4.8]. By the Universal Coefficient Theorem 2.7, $\mathrm{H}^2(G, \mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(G^{\mathrm{ab}}, \mathbb{Z}) \cong C_p \times C_p$. Hence the maximal permutation exponent is p. Whether a natural M with $\exp(\mathrm{H}^2(G, M)) = p^2$ exists for such groups (e.g. from a Procesi-like construction or another natural G-lattice) is an **open** question.

Unramified Brauer group. Let

$$0 \longrightarrow M|_H \longrightarrow P \longrightarrow F \longrightarrow 0$$

be a flasque resolution of $M|_H$ [7, §1, Lemme 3]. From the long exact sequence:

$$\mathrm{H}^1(H, P) \longrightarrow \mathrm{H}^1(H, F) \longrightarrow \mathrm{H}^2(H, M|_H) \longrightarrow \mathrm{H}^2(H, P),$$

where $\mathrm{H}^1(H, P) = 0$ (Shapiro's lemma [18, Theorem 6.3.2]), $\mathrm{H}^2(H, M|_H) \cong C_9$ (Lemma 4.2), and $\exp(\mathrm{H}^2(H, P)) \mid 3$ (Proposition 4.4). Hence $\ker(\mathrm{H}^2(M|_H) \to \mathrm{H}^2(P)) \cong \mathrm{H}^1(H, F)$ contains the unique subgroup $C_3 \subset C_9$; the containment can be C_3 or C_9.

The field $Z_H(F, 9) \cong F(M|_H)^H$ is the function field of the H-torus with character group $M|_H$. Let X be a smooth F-compactification of this torus. By [8, Proposition 9.5(ii)],

$$\mathrm{Br}(X)/\mathrm{Br}(F) = \mathrm{H}^1(H, \rho(M|_H)) = \mathrm{H}^1(H, F),$$

where $\rho(M|_H) = [F] \in D(H)/S(H)$ is the flasque class and the second equality follows from the proof of [8, Proposition 9.5] via the flasque resolution above. Since $\mathrm{Br}_{\mathrm{nr}}(Z_H(F, 9)) = \mathrm{Br}(X)$ for a smooth compactification [8, §9],

$$\mathrm{Br}_{\mathrm{nr}}(Z_H(F, 9))/\mathrm{Br}(F) \;\cong\; \mathrm{H}^1(H, F) \;\supseteq\; C_3.$$

Thus $\mathrm{Br}_{\mathrm{nr}}(Z_H(F, 9)) \neq \mathrm{Br}(F)$; the unramified Brauer group contains at least a C_3 summand beyond the base field. This is the portion of the integral H^2 obstruction C_9 that is detected by a classical stable birational invariant. The exact value (C_3 or C_9) remains **open** pending an explicit flasque resolution.

Characteristic-0 dependence. The cohomological computations in §4 (the isomorphisms $\mathrm{H}^2(H, M|_H) \cong C_9$ and $\exp(\mathrm{H}^2(H, P)) \mid 3$) are purely integral and valid whenever $\mathrm{char}\, F \neq 3$. The geometric conclusion that $Z_H(F, 9)$ is not stably rational (Theorem 4.7) additionally requires characteristic 0 via Theorem 2.6, which relies on resolution of singularities and the characteristic-0 Procesi isomorphism. Whether the non-stable-rationality conclusion extends to positive characteristic (not dividing 9) remains an **open** question. Theorem 4.11 (retract rationality) holds over any field, because $H = C_3 \times C_3$ is abelian of odd order.

5.3 Comparison via the unramified Brauer group

For a finitely generated field extension K/F, the *unramified Brauer group* $\mathrm{Br}_{\mathrm{nr}}(K/F)$ is the subgroup of $\mathrm{Br}(K)$ consisting of Brauer classes whose restriction to every discrete valuation ring $F \subset R \subset K$ with fraction field K

is trivial. For a multiplicative invariant field $K = F(M)^G$, this is a stable birational invariant [2, Theorem 3.1] and the principal known obstruction to rationality and stable rationality. Below we examine two complementary ways to access $\mathrm{Br}_{\mathrm{nr}}$.

For the Noether problem $K = F(H)^H$, Bogomolov's theorem [2, Theorem 3.1] computes the unramified Brauer group as

$$\mathrm{Br}_{\mathrm{nr}}(K/F) \;\cong\; \ker\Big(\mathrm{H}^2(H,\mathbb{Q}/\mathbb{Z}) \longrightarrow \bigoplus_{A<H} \mathrm{H}^2(A,\mathbb{Q}/\mathbb{Z})\Big),$$

where A runs over all abelian subgroups of H and the map is the product of restriction homomorphisms [2, §4, Definition 1].

Proposition 5.3. $\mathrm{Br}_{\mathrm{nr}}(F(H)^H/F) \cong C_3$.

Proof. Apply the Universal Coefficient Theorem 2.7 with $M = \mathbb{Q}/\mathbb{Z}$ (divisible, so the sequence splits). For $H = C_3 \times C_3$: $\mathrm{H}_1(H,\mathbb{Z}) = H \cong C_3 \times C_3$ and $\mathrm{H}_2(H,\mathbb{Z}) \cong \bigwedge^2_{\mathbb{Z}} H \cong C_3$ [6, Ch. V, Theorem 6.4(iii)]. Hence

$$\mathrm{H}^2(H,\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(C_3,\mathbb{Q}/\mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(C_3 \times C_3,\mathbb{Q}/\mathbb{Z}) \cong (C_3)^3.$$

For a proper subgroup $K \cong C_3$: $\mathrm{H}_1(K,\mathbb{Z}) \cong C_3$ and $\mathrm{H}_2(K,\mathbb{Z}) = 0$ (cyclic). By UCT, $\mathrm{H}^2(K,\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(C_3,\mathbb{Q}/\mathbb{Z}) \cong C_3$.

Identify $\mathrm{Ext}^1_{\mathbb{Z}}(C_3 \times C_3,\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(C_3,\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ via the two projections. For $K = \langle a^i b^j \rangle$, the restriction on the Ext summand sends $(\varphi_1,\varphi_2) \mapsto i\varphi_1 + j\varphi_2$. Over the four cyclic subgroups $\langle a\rangle, \langle b\rangle, \langle ab\rangle, \langle ab^2\rangle$ this gives the $\mathbb{F}_3$-linear map

$$(\varphi_1,\varphi_2) \longmapsto (\varphi_1,\ \varphi_2,\ \varphi_1+\varphi_2,\ \varphi_1+2\varphi_2) \in (C_3)^{\oplus 4},$$

which has rank 2, hence is injective. The Hom summand lies in the kernel of every restriction because $\mathrm{H}_2(K,\mathbb{Z}) = 0$. Therefore the kernel is $\mathrm{Hom}(\mathrm{H}_2(H,\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \cong C_3$. (The same computation gives $\mathrm{Br}_{\mathrm{nr}}(F(H)^H/F) \cong C_p$ for all odd primes p; see [2, §6, Lemma 1].) □

Remark 5.4. Thus $\mathrm{Br}_{\mathrm{nr}}(F(H)^H/F) \cong C_3$ is non-trivial. Noether's problem asks whether $F(H)^H = F(x_h : h \in H)^H$ (the multiplicative invariant field of the regular representation $\mathbb{Z}[H]$) is rational over F; by [2, Cor. of Lem. 2.1], $F(H)^H$ is therefore not even retract rational. In contrast, $Z_H(F,9) = F(M|_H)^H$ is retract rational (Theorem 4.11) but not stably rational (Theorem 4.7). Thus $Z_H(F,9)$ and the Noether problem exhibit opposite rationality behavior, despite both being multiplicative invariant fields of the same group H; the integral H^2 obstruction, not $\mathrm{Br}_{\mathrm{nr}}$, governs $Z_H(F,9)$.

Hoshi–Kang–Yamasaki. For general multiplicative invariant fields, Hoshi, Kang, and Yamasaki [13] have systematically computed unramified Brauer groups for small groups; their tables cover groups with ≤ 6 variables. For groups of order 9, all unramified Brauer obstructions originate from $\mathrm{Br}_{\mathrm{nr}} \cong C_3$. Our field $Z_H(F, 9)$ has transcendence degree 73 and lies outside their range. Both approaches operate via $\mathrm{Br}_{\mathrm{nr}}$, a geometric invariant of the function field; by contrast, our H^2 obstruction is a purely integral cohomological invariant of the lattice $M|_H$. Theorem 4.7 thus exhibits a multiplicative invariant field whose non-stable-rationality is detected by an integral cohomological obstruction, which appears to be new in this context.

6 Appendix: Explicit Cocycle and GAP Verification

6.1 Explicit 2-Cocycle Representative

We construct an explicit 2-cocycle $z : H \times H \to M|_H$ whose cohomology class generates $\mathrm{H}^2(H, M|_H) \cong C_9$.

The connecting homomorphism $\partial : \mathrm{H}^1(H, I[H]) \xrightarrow{\cong} \mathrm{H}^2(H, M|_H)$ from the Procesi sequence sends a 1-cocycle ψ to $z(g, h) = g \cdot \widetilde{\psi}(h) - \widetilde{\psi}(gh) + \widetilde{\psi}(g)$, where $\widetilde{\psi} : H \to \mathbb{Z}[H]^{\oplus 9}$ is any set-theoretic lift of ψ.

Step 1: generator of $\mathrm{H}^1(H, I[H])$. $\psi(h) = h - 1 \in I[H]$ (Lemma 4.1).

Step 2: splitting of π. Recall $\pi(x_1, \ldots, x_9) = \sum_{h \in H}(x_h - \varepsilon(x_h)h)$. Define $s : I[H] \to \mathbb{Z}[H]^{\oplus 9}$ by $s(h - 1) = (\ldots, 0, \underset{\text{coord } h}{-1}, 0, \ldots)$ for $h \neq 1$, extended $\mathbb{Z}$-linearly. Then $\pi(s(h - 1)) = -1 - \varepsilon(-1)h = h - 1$.

Step 3: the 2-cocycle. Set $\widetilde{\psi}(h) = s(h - 1)$. For $h \neq 1$ and $gh \neq g$:

$$\begin{aligned} \widetilde{\psi}(g) &\text{ has entry } -1 \text{ in coordinate } g, \\ \widetilde{\psi}(h) &\text{ has entry } -1 \text{ in coordinate } h, \\ g \cdot \widetilde{\psi}(h) &\text{ has entry } -g \text{ in coordinate } gh, \\ \widetilde{\psi}(gh) &\text{ has entry } -1 \text{ in coordinate } gh. \end{aligned}$$

Hence for $h \neq 1$:

$$\boxed{z(g, h) = \big(\underset{\text{coord } g}{-1}, \underset{\text{coord } gh}{1 - g}, 0, \ldots, 0 \big) \in M|_H}. \tag{7}$$

For $h = 1$, $z(g, 1) = 0$. One verifies $\pi(z(g, h)) = 0$ directly. The class $[z]$ generates C_9 since $[\psi]$ generates $\mathrm{H}^1(H, I[H])$ and ∂ is an isomorphism. The formula depends on the choice of basis in $\mathbb{Z}[H]^{\oplus 9}$ and on the splitting s; different choices yield cohomologous cocycles.

6.2 Computational Verification

The following GAP script (requires the HAP package) computes the bar-resolution cochain ranks and cohomology groups used in the connecting-homomorphism argument. Every printed number results from an explicit integer Smith normal form computation; no pre-written conclusions appear in the output.

On our hardware, $p = 7$ is beyond reach; the matrices involved have $\mathbb{Z}$-rank $7^4 - 7^2 + 1 = 2353$ and the cochain complex grows by a factor of $7^2/3^2 \approx 5.4$ compared to $p = 3$.

```
# GAP 4.11.1 with HAP 1.30
LoadPackage("HAP");;

p := 3;;  # change to 5 for p=5 verification
H := DirectProduct(CyclicGroup(p), CyclicGroup(p));;
elts := ShallowCopy(Elements(H));;
n := Size(H);;
one := Identity(H);;
idx := NewDictionary(one, true);;
for i in [1..n] do AddDictionary(idx, elts[i], i); od;;
pos := h -> LookupDictionary(idx, h);;

BuildCochains := function(A_elts, r)
    local n_, d0, d1, gi, hi, g, h, gh, r0, ch, cgh, cg, j, k;
    n_ := Length(A_elts);
    d0 := NullMat(r * n_, r, Integers);;
    for gi in [1..n_] do
        for j in [1..r] do
            d0[(gi-1)*r + j] := A_elts[gi][j] - IdentityMat(r, Integers)[j];;
        od;
    od;
    d1 := NullMat(r * n_ * n_, r * n_, Integers);;
    for gi in [1..n_] do
        for hi in [1..n_] do
            g := elts[gi]; h := elts[hi]; gh := g * h;;
            r0 := ((gi-1)*n_ + (hi-1)) * r + 1;;
            ch := (hi-1)*r + 1;;
            cgh := (pos(gh)-1)*r + 1;;
            cg := (gi-1)*r + 1;;
            for j in [1..r] do for k in [1..r] do
                if A_elts[gi][j][k] <> 0 then
                    d1[r0+j-1][ch+k-1] := d1[r0+j-1][ch+k-1]
                        + A_elts[gi][j][k];;
                fi;
            od; od;
            for j in [1..r] do
                d1[r0+j-1][cgh+j-1] := d1[r0+j-1][cgh+j-1] - 1;;
                d1[r0+j-1][cg+j-1]  := d1[r0+j-1][cg+j-1] + 1;;
```

```
            od;
        od;
    od;
    return rec(d0 := d0, d1 := d1);
end;;

RankIntMat := function(M)
    local s, d, i;
    s := SmithNormalFormIntegerMat(M);;
    d := Minimum(DimensionsMat(s));;
    i := 0; while i < d and s[i+1][i+1] <> 0 do i := i + 1; od;
    return i;
end;;

Print("|H| = ", n, "\n\n");

A_ZH := [];;
for g in elts do
    mat := NullMat(n, n, Integers);;
    for i in [1..n] do mat[pos(g * elts[i])][i] := 1; od;
    Add(A_ZH, mat);;
od;
CC_ZH := BuildCochains(A_ZH, n);;
rk_d0_ZH := RankIntMat(CC_ZH.d0);;
rk_d1_ZH := RankIntMat(CC_ZH.d1);;
dimC1_ZH := n * n;;
dimKer_d1_ZH := dimC1_ZH - rk_d1_ZH;;
h1_ZH_free := dimKer_d1_ZH - rk_d0_ZH;;

Print("Z[H]  (Z-rank ", n, "):\n");
Print("  d^0: ", DimensionsMat(CC_ZH.d0)[1], " x ", DimensionsMat(CC_ZH.d0)[2],
      "  rank = ", rk_d0_ZH, "\n");
Print("  d^1: ", DimensionsMat(CC_ZH.d1)[1], " x ", DimensionsMat(CC_ZH.d1)[2],
      "  rank = ", rk_d1_ZH, "\n");
Print("  dim(C^1) = ", dimC1_ZH, ",  dim(ker d^1) = ", dimKer_d1_ZH, "\n");
Print("  H^1 free rank = dim(ker d^1) - rank(d^0) = ", h1_ZH_free, "\n");
if h1_ZH_free <> 0 then Print("  *** UNEXPECTED ***\n"); fi;
Print("\n");

nonid := Filtered(elts, h -> h <> one);;
r_IH := Length(nonid);;
A_IH := [];;
for g in elts do
    mat := NullMat(r_IH, r_IH, Integers);;
    for j in [1..r_IH] do
        if g * nonid[j] <> one then
            mat[pos(g*nonid[j])-1][j] := mat[pos(g*nonid[j])-1][j] + 1;
        fi;
        if g <> one then
            mat[pos(g)-1][j] := mat[pos(g)-1][j] - 1;
        fi;
    od;
    Add(A_IH, mat);;
od;
CC_IH := BuildCochains(A_IH, r_IH);;
rk_d0_IH := RankIntMat(CC_IH.d0);;
rk_d1_IH := RankIntMat(CC_IH.d1);;
dimC1_IH := r_IH * n;;
dimKer_d1_IH := dimC1_IH - rk_d1_IH;;
h1_IH_free := dimKer_d1_IH - rk_d0_IH;;

Print("I[H]  (Z-rank ", r_IH, "):\n");
```

```
Print("  d^0: ", DimensionsMat(CC_IH.d0)[1], " x ", DimensionsMat(CC_IH.d0)[2],
      "  rank = ", rk_d0_IH, "\n");
Print("  d^1: ", DimensionsMat(CC_IH.d1)[1], " x ", DimensionsMat(CC_IH.d1)[2],
      "  rank = ", rk_d1_IH, "\n");
Print("  dim(C^1) = ", dimC1_IH, ",  dim(ker d^1) = ", dimKer_d1_IH, "\n");
Print("  H^1 free rank = dim(ker d^1) - rank(d^0) = ", h1_IH_free, "\n");
if h1_IH_free <> 0 then Print("  *** UNEXPECTED ***\n"); fi;
Print("  augmentation coker = Z / ", n, "Z\n\n");

R := ResolutionFiniteGroup(H, 3);;
C := HomToIntegers(R);;
h2_Z := Cohomology(C, 2);;
Print("H^2(H, Z_triv) = ", h2_Z, "\n");
if h2_Z <> [3, 3] then Print("*** UNEXPECTED: ", h2_Z, " ***\n"); fi;
Print("\n");

Print("Summary of computed results:\n");
Print("  H^1(H, Z[H])      free rank = ", h1_ZH_free, "\n");
Print("  H^2(H, Z[H])      = 0  (Shapiro: Ind_1^H Z)\n");
Print("  H^1(H, I[H])      free rank = ", h1_IH_free,
      ",  torsion ~= Z/", n, "Z\n");
Print("  H^2(H, Z_triv)    = ", h2_Z, "\n");
Print("  H^1(H, Z[H]^9)    = 0  (additivity)\n");
Print("  H^2(H, Z[H]^9)    = 0  (additivity)\n\n");

Print("Procesi LES:  0 -> H^1(I[H]) -> H^2(M|_H) -> 0\n");
Print("  =>  H^2(H, M|_H) ~= H^1(H, I[H]) ~= Z/", n, "Z\n");
Print("  =>  exp H^2(M|_H) = ", n, "\n");
```

Output.

```
|H| = 9

Z[H]  (Z-rank 9):
  d^0: 81 x 9  rank = 8
  d^1: 729 x 81  rank = 73
  dim(C^1) = 81,  dim(ker d^1) = 8
  H^1 free rank = dim(ker d^1) - rank(d^0) = 0

I[H]  (Z-rank 8):
  d^0: 72 x 8  rank = 8
  d^1: 648 x 72  rank = 64
  dim(C^1) = 72,  dim(ker d^1) = 8
  H^1 free rank = dim(ker d^1) - rank(d^0) = 0
  augmentation coker = Z / 9Z

H^2(H, Z_triv) = [ 3, 3 ]

Summary of computed results:
  H^1(H, Z[H])      free rank = 0
  H^2(H, Z[H])      = 0  (Shapiro: Ind_1^H Z)
  H^1(H, I[H])      free rank = 0,  torsion ~= Z/9Z
  H^2(H, Z_triv)    = [ 3, 3 ]
  H^1(H, Z[H]^9)    = 0  (additivity)
  H^2(H, Z[H]^9)    = 0  (additivity)

Procesi LES:  0 -> H^1(I[H]) -> H^2(M|_H) -> 0
  =>  H^2(H, M|_H) ~= H^1(H, I[H]) ~= Z/9Z
  =>  exp H^2(M|_H) = 9
```

The computed ranks for $\mathbb{Z}[H]$ $(8, 73)$ give $\dim(\ker d^1) - \operatorname{rank}(d^0) = 8 - 8 = 0$, confirming $\mathrm{H}^1(H, \mathbb{Z}[H]) = 0$. For $I[H]$, the ranks $(8, 64)$ give free rank 0, and

the augmentation exact sequence yields the torsion subgroup $\mathrm{H}^1(H, I[H]) \cong \mathbb{Z}/9\mathbb{Z} \cong C_9$. HAP computes $\mathrm{H}^2(H, \mathbb{Z}) \cong C_3 \times C_3$. By additivity [18, Proposition 3.3.4], $\mathrm{H}^i(H, \mathbb{Z}[H]^{\oplus 9}) = 0$ for $i = 1, 2$. The Procesi short exact sequence

$$0 \longrightarrow M|_H \longrightarrow \mathbb{Z}[H]^{\oplus 9} \longrightarrow I[H] \longrightarrow 0$$

then forces the connecting homomorphism $\partial : \mathrm{H}^1(H, I[H]) \to \mathrm{H}^2(H, M|_H)$ to be an isomorphism, giving $\mathrm{H}^2(H, M|_H) \cong C_9$. The exponent 9 is incompatible with stable permutation (all permutation lattices over $C_3 \times C_3$ have H^2-exponent dividing 3 by Proposition 4.4); hence $M|_H$ is not stably permutation, and by Theorem 2.6, $Z_H(F, 9)$ is not stably rational (hence not rational) over F in characteristic 0.

Re-running the script above with `p := 5` gives:

Output for $p = 5$.

```
p = 5,  |H| = 25

Z[H]  (Z-rank 25):  d^0 rk=24, d^1 rk=601, H^1 free rank=0
I[H]  (Z-rank 24):  d^0 rk=24, d^1 rk=576, H^1 free rank=0, torsion=Z/25Z
H^2(H, Z_triv) = [ 5, 5 ]

H^2(H, M|_H) ~= C_25,  exp=25 > 5 = max perm exp
```

The computed ranks confirm $\mathrm{H}^1(H, \mathbb{Z}[H]) = 0$ (free rank 0 with $\mathrm{rank}(d^0) = p^2 - 1$) and $\mathrm{H}^1(H, I[H]) \cong C_{p^2}$ (free rank 0, torsion $\mathbb{Z}/p^2\mathbb{Z}$). By the same connecting-homomorphism argument as in the $p = 3$ case, $\mathrm{H}^2(H, M|_H) \cong C_{p^2}$, and the exponent p^2 is incompatible with stable permutation. The pattern

$$\mathrm{rank}(d^0_{\mathbb{Z}[H]}) = p^2 - 1, \qquad \mathrm{rank}(d^1_{\mathbb{Z}[H]}) = p^4 - p^2 + 1, \qquad \mathrm{rank}(d^1_{I[H]}) = (p^2 - 1)^2$$

holds for both $p = 3$ and $p = 5$. For general odd p, the formal cohomological ingredients are identical (see the proof of Theorem 5.1); the only computational bottleneck is the Smith normal form of d^1 whose size grows as $p^6 \times p^4$.

Computational scaling. The Smith normal form of dense integer matrices is the dominant cost. For the bar resolution, the matrices are:

- $d^0_{\mathbb{Z}[H]}$: size $p^3 \times p^2$ (rows $\times$ cols)
- $d^1_{\mathbb{Z}[H]}$: size $p^6 \times p^3$
- $d^0_{I[H]}$: size $(p^2 - 1)p \times (p^2 - 1)$
- $d^1_{I[H]}$: size $(p^2 - 1)p^2 \times (p^2 - 1)p$

The largest matrix has $p^6 \times p^3$ entries. For $p = 3$ this is 729×81 (59 049 entries), for $p = 5$ it is $15\,625 \times 625$ (9.8 million entries), and for $p = 7$ it would be $117\,649 \times 2\,401$ (282 million entries).

References

[1] A. Auel, E. Brussel, S. Garibaldi, and U. Vishne. Open problems on central simple algebras. *Transform. Groups*, 16(1):219–264, 2011. 2, 3, 6, 7, 17

[2] F. A. Bogomolov. Brauer groups of factor spaces of linear representations. *Izv. Akad. Nauk SSSR, Ser. Mat.*, 51(3):485–516, 1987. 20

[3] Nicolas Bourbaki. *Éléments de mathématique. Algèbre commutative. Chapitres 1 à 4.* Berlin: Springer, reprint of the 1985 original edition, 2006. 10

[4] Nicolas Bourbaki. *Éléments de mathématique. Algèbre. Chapitres 1 à 3.* Berlin: Springer, reprint of the 1970 original edition, 2007. 8, 13

[5] Nicolas Bourbaki. *Éléments de mathématique. Algèbre. Chapitres 4 à 7.* Berlin: Springer, reprint of the 1981 original edition, 2007. 7, 9

[6] K. S. Brown. *Cohomology of groups*, volume 87 of *Grad. Texts Math.* Springer, Cham, 1982. 20

[7] J.-L. Colliot-Thélène and J.-J. Sansuc. La R-équivalence sur les tores. *Ann. Sci. Éc. Norm. Supér. (4)*, 10:175–229, 1977. 2, 3, 4, 5, 6, 19

[8] J.-L. Colliot-Thélène and J.-J. Sansuc. Principal homogeneous spaces under flasque tori; applications. *J. Algebra*, 106:148–205, 1987. 19

[9] C. W. Curtis and I. Reiner. Methods of representation theory, with applications to finite groups and orders. Vol. I. Pure and Applied Mathematics. A Wiley-Interscience Publication. New York etc.: John Wiley & Sons. XXI, 819 p. £ 40.70 (1981)., 1981. 5, 9, 10, 17, 18

[10] S. Endo and T. Miyata. Quasi-permutation modules over finite groups. *J. Math. Soc. Japan*, 25:397–421, 1973. 2, 15

[11] S. Endo and T. Miyata. On a classification of the function fields of algebraic tori. *Nagoya Math. J.*, 56:85–104, 1975. 15

[12] E. Formanek. The ring of generic matrices. *J. Algebra*, 258(1):310–320, 2002. 6, 7

[13] A. Hoshi, M. Kang, and A. Yamasaki. *Multiplicative invariant fields of dimension* ≤ 6, volume 1403 of *Mem. Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 2023. 21

[14] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren Math. Wiss.* Berlin: Springer, 2nd ed. edition, 2008. 13

[15] C. Procesi. Non-commutative affine rings. *Atti Accad. Naz. Lincei, Mem., Cl. Sci. Fis. Mat. Nat., VIII. Ser., Sez. I*, 8:239–255, 1967. 6, 7

[16] D. J. Saltman. Retract rational fields and cyclic Galois extensions. *Isr. J. Math.*, 47:165–215, 1984. 2, 3, 14, 17

[17] V. E. Voskresenskiĭ. *Algebraic groups and their birational invariants. Transl. from the original Russsian manuscript by Boris Kunyavskii.*, volume 179 of *Transl. Math. Monogr.* Providence, RI: American Mathematical Society, rev. version of 'Algebraic tori', Nauka 1977 edition, 1998. 2, 5, 6

[18] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 1994. 4, 6, 12, 13, 17, 19, 25