

COMMON CYCLIC SUBFIELDS OF DEGREE- p DIVISION ALGEBRAS OVER COMPLETE DISCRETE VALUATION FIELDS

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ABSTRACT. Let A, B be degree- p division algebras over a complete discrete valuation field with $\text{ind}(A \otimes B) < \text{ind}(A) \cdot \text{ind}(B)$. We study the existence of common cyclic maximal subfields of A and B in two complementary settings. In characteristic $\neq p$ (with $\mu_p \subset F$), we classify the common subfields via the Faddeev exact sequence on the Brauer group, giving explicit criteria in the unramified, mixed, and both-ramified valuation regimes, and prove a norm intersection formula. In characteristic p , we prove that when the residue field has trivial p -torsion Brauer group, every pair of degree- p p -algebras shares the common Artin-Schreier splitting field $K(\wp^{-1}(\pi))$. For general non-Henselian fields, we identify structural obstructions in both settings.

1. INTRODUCTION

Let F be a field and let A, B be F -central division algebras. Problem 9.2 of [1] asks: if $\text{ind}(A \otimes B) < \text{ind}(A) \cdot \text{ind}(B)$, what are the common subfields of A and B ? We study this problem for degree- p algebras over complete discrete valuation fields, in two complementary settings: fields of characteristic $\neq p$ containing a primitive p -th root of unity, and fields of characteristic p .

The problem has a natural stratification by the index of the tensor product. When $\text{ind}(A \otimes B) = 1$ (the split case), A and B are opposites in the Brauer group and a common subfield always exists (Theorem 3.1). When $\text{ind}(A \otimes B) = p$ (the non-split index-drop case), we build the classification in three layers.

Layer 1 — field-theoretic constraints. The dimension $d_F = \dim_{\mathbb{F}_p} F^\times / F^{\times p}$ controls the size of the Brauer group. When $d_F \leq 2$, any two non-split symbols are $\mathbb{F}_p$ -multiples and share all splitting fields (Theorem 3.2).

Layer 2 — Brauer class dependence. Even when $d_F > 2$, if $[A] = \lambda[B]$ in $\text{Br}(F)[p]$, then both lie in the cyclic subgroup generated by $[A \otimes B]$, and a common subfield exists (Theorem 4.1).

Layer 3 — CDVF structure (characteristic $\neq p$). Over $F = K((t))$ with $\text{char}(K) \neq p$ and $\mu_p \subset F$, every degree- p division algebra is either unramified or nicely semiramified ($\cong (t, \beta)_F$). The Faddeev exact sequence reduces the splitting condition $[A] \in \text{im}(c \cup -)$ to matching tame symbols and inertial classes. In the unramified case, common subfields correspond to $\bar{u}$ with both inertial classes in $\text{im}(\bar{u} \cup -)$; in the mixed case, they exist iff $\bar{c} = \bar{b}_1$ and $[\bar{A}_2] \in \text{im}(\bar{b}_1 \cup -)$; in the both-ramified case, ramified common subfields

always exist, while unramified ones require $\bar{\beta}_1 = \bar{\beta}_2$ (Propositions 5.1–5.5). We also prove the norm intersection formula (Proposition 5.6).

Layer 4 — characteristic p . Over a CDVF $K = \kappa((\pi))$ of characteristic p , when $\text{Br}(\kappa)[p] = 0$ the tame symbol is an isomorphism and every p -algebra is of the form $[\pi, \gamma]_{p,K}$. Consequently $K(\wp^{-1}(\pi))$ is a common splitting field for *all* pairs of degree- p p -algebras, without any hypothesis on $\text{ind}(A \otimes B)$ (Theorem 6.1); the index-drop condition follows a posteriori. When $\text{Br}(\kappa)[p] \neq 0$, the unramified component obstructs the common splitting field (Remark 6.3).

Beyond CDVFs. For general fields, we identify two structural obstructions in characteristic $\neq p$, and the characteristic p case remains open for non-Henselian fields (Section 7).

Relation to known results. Albert [2] solved the $p = 2$ case. Risman [14] classified common subfields over local and global fields. Chapman [4] developed linkage theory for p -algebras, and Chapman–Srimathy [6] proved the $\text{TRPI} \Leftrightarrow \text{TRC}$ equivalence over CDVFs. Jacob–Wadsworth [9] constructed CDVF counterexamples for any odd p in characteristic $\neq p$. Chapman [5] studied symbol changes in characteristic p , and CFM [3] related common splitting fields to uniform Witt vector presentations. Our work complements these results by giving explicit residue conditions for both settings.

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2. PRELIMINARIES

2.1. Characteristic $\neq p$: Kummer theory. Throughout this subsection, p is an odd prime and F is a field of characteristic $\neq p$ containing a primitive p -th root of unity ζ_p . For $a, b \in F^\times$, the symbol algebra $(a, b)_p$ is the F -algebra generated by x, y with $x^p = a$, $y^p = b$, $yx = \zeta_p xy$. It is a central simple algebra of degree p .

2.1.1. The Merkurjev–Suslin theorem.

Theorem 2.1 (Merkurjev–Suslin [13]). *The cup product induces a surjection $\psi : \wedge^2(F^\times/F^{\times p}) \twoheadrightarrow \text{Br}(F)[p]$. Every class in $\text{Br}(F)[p]$ is represented by a symbol algebra of degree p .*

The Kummer isomorphism identifies $F^\times/F^{\times p} \cong H^1(F, \mu_p)$. The cup product $\cup : H^1(F, \mu_p) \times H^1(F, \mu_p) \rightarrow H^2(F, \mu_p) \cong \text{Br}(F)[p]$ corresponds to the symbol map and is alternating for odd p .

2.1.2. CDVF structure (characteristic $\neq p$). Let $F = K((t))$ be a complete discretely valued field with uniformizer t , valuation ring $\mathcal{O}_F$, and residue field K of characteristic $\neq p$. The tame symbol exact sequence [7] is

$$0 \longrightarrow \mathrm{Br}(K)[p] \longrightarrow \mathrm{Br}(F)[p] \xrightarrow{\partial} K^\times / K^{\times p} \longrightarrow 0.$$

Theorem 2.2 (Jacob–Wadsworth [8], Theorem 5.15). *Every tame division algebra D over a complete DVF F decomposes as $D \cong I \otimes_F N$, where I is inertial (unramified) and N is nicely semiramified (NSR). Moreover $\mathrm{ind}(D) = \mathrm{ind}(I) \cdot \mathrm{ind}(N)$ and $\mathrm{per}(D) = \mathrm{lcm}(\mathrm{per}(I), \mathrm{per}(N))$.*

2.2. Characteristic p : Artin–Schreier–Witt theory. Let F be a field of characteristic $p > 0$. The Artin–Schreier map $\wp : F \rightarrow F$ is defined by $\wp(x) = x^p - x$. The additive group $F/\wp(F)$ parametrises the cyclic Galois extensions of F of degree p : to $\alpha \in F$, we associate $F(\wp^{-1}(\alpha))$.

2.2.1. ASW symbols. For $\alpha, \beta \in F$, the ASW symbol algebra $[\alpha, \beta]_{p,F}$ is the F -algebra generated by x, y with relations

$$x^p - x = \alpha, \quad y^p = \beta, \quad yxy^{-1} = x + 1.$$

It is a central simple F -algebra of degree p . Every central simple F -algebra of degree p and period dividing p is Brauer equivalent to an ASW symbol.

The cup product pairing in characteristic p is [10, 7]

$$\cup : F/\wp(F) \times F/\wp(F) \longrightarrow \mathrm{Br}(F)[p],$$

mapping (α, β) to the Brauer class of $[\alpha, \beta]_{p,F}$. The pairing is bi-additive and alternating: $[\alpha, \alpha]_{p,F}$ is split for all α .

2.2.2. Artin–Schreier splitting criterion. An AS extension $L = F(\wp^{-1}(\omega))$ splits $[\alpha, \beta]_{p,F}$ if and only if ω belongs to the $\mathbb{F}_p$ -span of $\{\alpha, \beta\}$ modulo $\wp(F)$ [5]. Explicitly, $[\alpha + \beta, \beta]_{p,F} = [\alpha, \beta]_{p,F}$ and $[\alpha + \wp(v), \beta]_{p,F} = [\alpha, \beta]_{p,F}$. Consequently, $[\alpha, \beta]_{p,F}$ is split by exactly the p extensions $F(\wp^{-1}(\lambda\alpha + \mu\beta))$ for $(\lambda, \mu) \in \mathbb{F}_p^2 \setminus \{(0, 0)\}$.

2.2.3. CDVFs in characteristic p . For a CDVF $K = \kappa((\pi))$ of characteristic p with residue field κ , Kato [10] (Theorem 3(2)) established the tame symbol exact sequence:

$$0 \longrightarrow \mathrm{Br}(\kappa)[p] \longrightarrow \mathrm{Br}(K)[p] \xrightarrow{\partial} \kappa/\wp(\kappa) \longrightarrow 0.$$

For an algebra $A = [\alpha, \beta]$ where $v(\alpha) = 1$, $v(\beta) = 0$, the tame symbol is $\partial([A]) = \bar{\beta} \in \kappa/\wp(\kappa)$.

3. THE SPLIT CASE AND SMALL FIELDS

3.1. Split tensor product.

Theorem 3.1. *Let $A = (a, b)_p$, $B = (c, d)_p$ be degree- p symbol algebras over F with $\mu_p \subset F$, $\mathrm{ind}(A) = \mathrm{ind}(B) = p$, and $\mathrm{ind}(A \otimes B) = 1$. Then A and B contain a common cyclic maximal subfield of degree p .*

Proof. Since $\text{ind}(A \otimes B) = 1$, we have $[B] = -[A]$ in $\text{Br}(F)[p]$. By Theorem 2.1, write $[A] = (e, f)_p$. Then $[B] = -(e, f)_p = (e, f^{-1})_p$. Both $(e, f)_p$ and $(e, f^{-1})_p$ have degree p and Brauer class matching A, B respectively, hence are isomorphic to them as F -algebras (two central simple algebras of equal degree are isomorphic iff they are Brauer equivalent). The field $F(\sqrt[p]{e})$ splits both, hence by K  the's theorem embeds as a common maximal subfield. $\square$

3.2. Small Kummer dimension.

Theorem 3.2. *Let F be a field with $\mu_p \subset F$, $\text{char}(F) \neq p$, and $\dim_{\mathbb{F}_p} F^\times / F^{\times p} = 2$. Let $A = (a, b)_p$, $B = (c, d)_p$ be non-split symbol algebras with $F(\sqrt[p]{a}) \neq F(\sqrt[p]{c})$. Then A and B share a common cyclic splitting field of degree p .*

Proof. Let $V = F^\times / F^{\times p}$. Since $\dim V = 2$, $\dim \wedge^2 V = 1$. By Theorem 2.1, $\dim_{\mathbb{F}_p} \text{Br}(F)[p] \leq 1$, and non-split A forces $\dim \text{Br}(F)[p] = 1$. Since $F(\sqrt[p]{a}) \neq F(\sqrt[p]{c})$, the classes of a, c form a basis of V . Let $\varepsilon = (a, c)_p$; then ε generates $\text{Br}(F)[p]$ and $[A], [B]$ are powers of ε . The field $L = F(\sqrt[p]{a})$ splits ε (since $\text{res}_L(a) = 0$), hence splits both A and B . $\square$

Remark 3.3. For $\dim V = 0$, $\text{Br}(F)[p] = 0$ and there are no non-split algebras. For $\dim V = 1$, any two non-split symbols are Brauer-equivalent: if $\text{ind}(A \otimes B) = 1$, Theorem 3.1 applies; if $\text{ind}(A \otimes B) = p$, Theorem 4.1 applies.

4. DEPENDENT BRAUER CLASSES

Theorem 4.1. *Let $A = (a, b)_p$, $B = (c, d)_p$ be degree- p division algebras over F with $\text{ind}(A \otimes B) = p$. If $[A] = \lambda[B]$ in $\text{Br}(F)[p]$ for some $\lambda \in \mathbb{F}_p^\times$, then A and B share a common cyclic maximal subfield.*

Proof. Set $E = A \otimes B$, so $\text{ind}(E) = p$ and $\text{per}(E) = p$. Since $\text{ind}(E) = p \neq 1$, $[E] = [A] + [B] \neq 0$, hence $\lambda \neq -1$ and $[E] = (1 + \lambda)[B] \neq 0$. Thus $[B] = (1 + \lambda)^{-1}[E]$ and $[A] = \lambda(1 + \lambda)^{-1}[E]$; both lie in the cyclic subgroup generated by $[E]$.

By Theorem 2.1, write $[E] = (e, f)_p$. The field $L = F(\sqrt[p]{e})$ splits $(e, f)_p$ and all its powers, hence splits both A and B . By K  the's theorem, L embeds as a common maximal subfield. $\square$

Remark 4.2. Theorem 3.2 (small Kummer dimension) is a special case: when $\dim_{\mathbb{F}_p} F^\times / F^{\times p} \leq 2$, the Brauer group $\text{Br}(F)[p]$ has $\mathbb{F}_p$ -dimension at most 1, so any two non-split symbols automatically satisfy $[A] = \lambda[B]$, and Theorem 4.1 applies.

5. CDVF CLASSIFICATION IN CHARACTERISTIC $\neq p$

We now analyse common subfields over $F = K((t))$ using the Faddeev exact sequence for the Brauer group. Set $A = (a_1, b_1)_F$, $B = (a_2, b_2)_F$.

5.1. The splitting condition over a CDVF. For a symbol algebra $A = (a, b)_p$, a degree- p cyclic extension $F(\sqrt[p]{c})$ splits A iff $[A]$ lies in the image of the cup product map $c \cup -: H^1(F, \mu_p) \rightarrow H^2(F, \mu_p) \cong \text{Br}(F)[p]$. Equivalently, there exists $x \in F^\times$ such that

$$(1) \quad [A] = c \cup x \quad \text{in } H^2(F, \mu_p).$$

(Indeed, $F(\sqrt[p]{c})$ splits A iff the restriction of $[A]$ to $F(\sqrt[p]{c})$ vanishes; the kernel of restriction is precisely $\text{im}(c \cup -)$ by the Merkurjev–Suslin theorem.)

Over the CDVF F , the Faddeev exact sequence $0 \rightarrow H^2(K, \mu_p) \rightarrow H^2(F, \mu_p) \xrightarrow{\partial} H^1(K, \mu_p) \rightarrow 0$ decomposes each Brauer class into its tame symbol and its inertial class. Two classes in $H^2(F)$ coincide iff both components coincide. Hence (1) is equivalent to the pair of equations

$$(2) \quad \partial([A]) = \partial(c \cup x) \quad \text{in } H^1(K, \mu_p),$$

$$(3) \quad [\bar{A}] = s_2(c \cup x) \quad \text{in } H^2(K, \mu_p),$$

where $[\bar{A}]$ is the inertial class of A and s_2 is the specialization map on H^2 .

Write $c = t^i u$, $x = t^j y$ with $0 \leq i, j < p$ and $u, y \in \mathcal{O}_F^\times$. The tame symbol formula for symbol algebras [7] is $\partial((\alpha, \beta)_p) = (-1)^{v(\alpha)v(\beta)}(v(\beta)\bar{\alpha} - v(\alpha)\bar{\beta})$. For odd p , multiplication by -1 is an automorphism of $H^1(K, \mu_p)$, so the sign may be absorbed into the choice of representatives.¹ We therefore work with the simplified form

$$(4) \quad \partial(c \cup x) = j \bar{u} - i \bar{y} \quad \text{in } H^1(K, \mu_p).$$

The inertial class $s_2(c \cup x)$ depends on the valuations i, j . By the JW decomposition [8] (or GS2017, Corollary 6.8.3), a symbol algebra with exactly one ramified entry is nicely semiramified and has trivial inertial class; an unramified or both-ramified symbol carries the cup product of the unit residues. Hence:

- If $i = j = 0$ (both unramified): $s_2(c \cup x) = \bar{u} \cup \bar{y}$.
- If $i = 1, j = 0$ or $i = 0, j = 1$ (mixed): $s_2(c \cup x) = 0$.
- If $i = j = 1$ (both ramified): $s_2(c \cup x) = \bar{u} \cup \bar{y}$.

For a degree- p symbol algebra $A = (a, b)_p$ with $\text{ind}(A) = p$, the HNW index formula gives

$$(5) \quad \text{ind}(A) = \text{per}(\partial([A])) \cdot \text{ind}([\bar{A}]).$$

Consequently $[\bar{A}] = 0$ whenever $\partial([A]) \neq 0$, and $[\bar{A}] \neq 0$ with $\text{ind}([\bar{A}]) = p$ whenever $\partial([A]) = 0$.

Write $a_j = t^{v_j} \alpha_j$, $b_j = t^{w_j} \beta_j$ with α_j, β_j units. By the Jacob–Wadsworth decomposition [8], every degree- p division algebra A over $F = K((t))$ with $\text{char}(K) \neq p$ and $\mu_p \subset F$ is either unramified ($\partial([A]) = 0$) or nicely semiramified ($\partial([A]) \neq 0$). In the unramified case $A \cong (\alpha, \beta)_F$ with $\alpha, \beta \in \mathcal{O}_F^\times$. In the nicely semiramified case $A \cong (t, \beta)_F$ with $\beta \in \mathcal{O}_F^\times$ (see JW, Theorem 5.15 and the discussion of NSR algebras of degree p). Thus, after choosing suitable symbol presentations, we may assume $v(b_1) = v(b_2) = 0$ and $v(a_j) \in \{0, 1\}$. The pair $(v(a_1), v(a_2))$ then falls into one of three types analysed below.

¹In the both-ramified case ($i = j = 1$), the sign $(-1)^{1 \cdot 1} = -1$ appears; multiplying the auxiliary element $\bar{y}$ by -1 absorbs this sign without affecting the solution space.

5.2. Unramified case.

Proposition 5.1. *Assume $v(a_1) = v(a_2) = 0$. Then $F(\sqrt[p]{c})$ is a common subfield of A and B iff c is an unramified unit and the residue classes satisfy $[\bar{A}_1], [\bar{A}_2] \in \text{im}(\bar{c} \cup -)$ in $H^2(K, \mu_p)$.*

Proof. Since $\partial([A_j]) = 0$, (2) reads $j\bar{u} - i\bar{y} = 0$. If $i = v(c) \neq 0$, either $j = 0$ (giving $\bar{y} = 0$, whence $[c \cup x]$ has trivial inertial class $\neq [\bar{A}_j]$) or $j \neq 0$ (giving $s_2(c \cup x) = \bar{u} \cup \bar{y} = 0$, again trivial). Hence $i = 0$: $c = u$ is a unit.

With $i = 0$, (4) gives $j\bar{u} = 0$. Since $\bar{u} \neq 0$ (otherwise $c \in F^{\times p}$), $j = 0$ and $x = y$ is a unit. Then $s_2(u \cup y) = \bar{u} \cup \bar{y}$. By (3), $[\bar{A}_j] = \bar{u} \cup \bar{y}_j$ for some $\bar{y}_j$, i.e. $[\bar{A}_j] \in \text{im}(\bar{u} \cup -)$ for $j = 1, 2$. $\square$

Remark 5.2. The condition $[\bar{A}_1], [\bar{A}_2] \in \text{im}(\bar{u} \cup -)$ forces $[\bar{A}_1]$ and $[\bar{A}_2]$ to be $\mathbb{F}_p$ -multiples of each other in $H^2(K, \mu_p)$, since $\text{im}(\bar{u} \cup -)$ is 1-dimensional. Hence common unramified subfields exist precisely when the residue algebras have dependent Brauer classes—the situation of Theorem 4.1.

5.3. Mixed case.

Proposition 5.3. *Assume $v(a_1) = 1, v(a_2) = 0$ (after normalisation). Then $F(\sqrt[p]{c})$ is a common subfield iff $\bar{c} = \bar{b}_1$ (up to an $\mathbb{F}_p^\times$ -scalar) and $[\bar{A}_2] \in \text{im}(\bar{b}_1 \cup -)$ in $H^2(K, \mu_p)$.*

Proof. For $A_1 = (t\alpha_1, \beta_1)$: $\partial([A_1]) = \bar{\beta}_1$, $[\bar{A}_1] = 0$. Equation (2): $j\bar{u} - i\bar{y} = \bar{\beta}_1$.

If $i = 0$ (so $c = u$): $j\bar{u} = \bar{\beta}_1 \neq 0$, thus $j \neq 0$ and $\bar{u} = j^{-1}\bar{\beta}_1$. Replacing c by a suitable p -th power multiple within its Kummer class, we may assume $\bar{u} = \bar{\beta}_1$. Then $s_2(u \cup ty) = 0$, matching $[\bar{A}_1] = 0$.

For $A_2 = (\alpha_2, \beta_2)$: $\partial([A_2]) = 0$, $[\bar{A}_2] \neq 0$. Take x_2 unramified ($j = 0$): $\partial(u \cup x_2) = 0$ (match). $s_2(u \cup x_2) = \bar{u} \cup \bar{x}_2 = \bar{\beta}_1 \cup \bar{x}_2$. Need $\bar{\beta}_1 \cup \bar{x}_2 = [\bar{A}_2]$, solvable iff $[\bar{A}_2] \in \text{im}(\bar{\beta}_1 \cup -)$.

If $i \neq 0$ (ramified c , say $i = 1$): to match $\partial([A_2]) = 0$, we need $\partial(c \cup x_2) = 0$, which forces either x_2 unramified ($j = 0$, $\bar{y}_2 = 0$, giving trivial inertial class) or x_2 ramified ($j = 1$, $\bar{y}_2 = \bar{u}$, giving $s_2(c \cup x_2) = \bar{u} \cup \bar{u} = 0$). In both cases the inertial class is $0 \neq [\bar{A}_2]$, so no solution exists. $\square$

Remark 5.4. When $[\bar{A}_2] \in \text{im}(\bar{\beta}_1 \cup -)$, the common subfield is $F(\sqrt[p]{u})$ with $\bar{u} = \bar{\beta}_1$. Generically (when $d_K \geq 3$), $\bar{\beta}_1$ and $[\bar{A}_2]$ are independent and no common subfield exists.

5.4. Both-ramified case.

Proposition 5.5. *Assume $v(a_1) = v(a_2) = 1$. After normalisation, $A_1 \cong (t, \beta_1)_F$, $A_2 \cong (t, \beta_2)_F$.*

- **Ramified** common subfields $F(\sqrt[p]{tu})$ exist for all $\bar{u} \in H^1(K, \mu_p)$, $\bar{u} \neq 0$, regardless of $\bar{\beta}_1, \bar{\beta}_2$.
- **Unramified** common subfields $F(\sqrt[p]{u})$ exist iff $\bar{\beta}_1 = \bar{\beta}_2$ in $H^1(K, \mu_p)$, in which case $\bar{u} = \bar{\beta}_1$.

Proof. For ramified $c = tu$ ($i = 1$): take x_j unramified ($j = 0$) with $\bar{y}_j = -\bar{\beta}_j$. Then $\partial(tu \cup x_j) = -\bar{y}_j = \bar{\beta}_j = \partial([A_j])$ and $s_2(tu \cup x_j) = 0 = [\bar{A}_j]$. Hence $[A_j] = tu \cup x_j$ for both $j = 1, 2$, using *different* auxiliary elements x_1, x_2 .

For unramified $c = u$ ($i = 0$): for $j = 1, 2$, take $x_j = ty_j$. Then $\partial(u \cup ty_j) = \bar{u}$ must equal $\bar{\beta}_j$, forcing $\bar{u} = \bar{\beta}_1 = \bar{\beta}_2$. The inertial condition $s_2(u \cup ty_j) = 0$ matches $[A_j] = 0$ automatically. $\square$

5.5. Norm intersection formula.

Proposition 5.6. *Let $m_j = F(\sqrt[p]{a_j})$. Over $F = K((t))$ with $\text{char}(K) \neq p$ and $\mu_p \subset F$, we have $N_{m_1/F}(m_1^\times) F^{\times p} \cap N_{m_2/F}(m_2^\times) F^{\times p} = F^{\times p}$ in the mixed case, and in the both-ramified case when $\bar{\beta}_1 \neq \bar{\beta}_2$. If $\bar{\beta}_1 = \bar{\beta}_2$, the intersection contains $F(\sqrt[p]{t})$ and is strictly larger.*

Proof. An element $x \in F^\times$ lies in $N(m_j^\times) F^{\times p}$ iff $(x, a_j)_p = 0$ in $\text{Br}(F)[p]$. Write $x = t^i u$.

For the mixed case ($a_1 = t\alpha_1$, $a_2 = \alpha_2$): $\partial((x, \alpha_2)_p) = -i\bar{\alpha}_2$ forces $i = 0$ (since $\bar{\alpha}_2 \neq 0$). Then $\partial((u, t\alpha_1)_p) = \bar{u}$ forces $\bar{u} = 0$, so $u \in F^{\times p}$ by Hensel's lemma. The inertial conditions are automatic when $\bar{u} = 0$. Hence $x = u \in F^{\times p}$.

For the both-ramified case ($a_j = t\alpha_j$): $\partial((x, t\alpha_j)_p) = (-1)^i(\bar{u} - i\bar{\alpha}_j)$. Both vanishing gives $\bar{u} = i\bar{\alpha}_1 = i\bar{\alpha}_2$, so $i(\bar{\alpha}_1 - \bar{\alpha}_2) = 0$. When $\bar{\alpha}_1 \neq \bar{\alpha}_2$, $i = 0$ and $\bar{u} = 0$, so $x \in F^{\times p}$ (the inertial class of $(x, t\alpha_j)_p$ is $\bar{u} \cup \bar{\alpha}_j$ for $i \neq 0$, which vanishes when $\bar{u} = i\bar{\alpha}_j$ by the alternating property; for $i = 0$ the inertial class is trivial). When $\bar{\alpha}_1 = \bar{\alpha}_2$, any $x = t^i u$ with $\bar{u} = i\bar{\alpha}_1$ lies in the intersection, giving a subgroup of dimension at least 1. $\square$

Remark 5.7. In the unramified case, the norm intersection problem reduces to the residue field K , where Merkurjev's analysis [12] applies.

6. A CDVF THEOREM IN CHARACTERISTIC p

Every degree- p division algebra over K is Brauer equivalent to an ASW symbol $[\alpha, \beta]_{p,K}$ (the characteristic- p analogue of the Merkurjev–Suslin theorem [10]). A common splitting field for the Brauer classes is also a common splitting field for the division algebras themselves, since a degree- p extension splits a Brauer class iff it embeds as a maximal subfield of the corresponding division algebra.

Theorem 6.1 (CDVF EDCS in characteristic p). *Let $K = \kappa((\pi))$ be a CDVF of characteristic p with $\text{Br}(\kappa)[p] = 0$. Let A, B be degree- p division algebras over K . Then $K(\wp^{-1}(\pi))$ is a common Artin–Schreier splitting field of degree p for both A and B . (In particular $\text{ind}(A \otimes B) < p^2$ follows from the existence of a common maximal subfield.)*

Proof. The tame symbol exact sequence for K (Kato [10], Theorem 3(2)) is

$$0 \longrightarrow \text{Br}(\kappa)[p] \longrightarrow \text{Br}(K)[p] \xrightarrow{\partial} \kappa/\wp(\kappa) \longrightarrow 0.$$

Since $\text{Br}(\kappa)[p] = 0$ by hypothesis, ∂ is an isomorphism. For any class $[A] \in \text{Br}(K)[p]$, let $\bar{\gamma} = \partial([A]) \in \kappa/\wp(\kappa)$ and lift γ to an element of $\mathcal{O}_K$ with $v(\gamma) \geq 0$. The ASW symbol $[\pi, \gamma]_{p,K}$ then has tame symbol $\bar{\gamma}$ and index p (since its tame symbol is non-zero whenever $\bar{\gamma} \neq 0$; if $\bar{\gamma} = 0$ then $[A]$ is unramified and already lies in $\text{Br}(\kappa)[p] = 0$). Thus $[A] - [\pi, \gamma]_{p,K} \in \ker(\partial) = \text{Br}(\kappa)[p] = 0$. Hence $[A] = [\pi, \gamma_1]_{p,K}$ and similarly $[B] = [\pi, \gamma_2]_{p,K}$ for some $\gamma_1, \gamma_2 \in K$.

Let $L = K(\wp^{-1}(\pi))$. The kernel of the restriction map $\text{res}_{L/K} : \text{Br}(K)[p] \rightarrow \text{Br}(L)[p]$ is precisely the image of $\pi \cup - : H^1(K, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2(K, \mathbb{Z}/p\mathbb{Z})$ [10]. Since $[\pi, \gamma_j]_{p,K} = \pi \cup \gamma_j$ lies in this image, we have $\text{res}_{L/K}([A]) = \text{res}_{L/K}([B]) = 0$. Thus L splits both A and B . $\square$

Example 6.2. The theorem applies when $\kappa = \mathbb{F}_p$ (finite field, $\text{Br}(\mathbb{F}_p)[p] = 0$), $\kappa = \bar{\mathbb{F}}_p$ (algebraically closed), or any field of $\text{cd}_p(\kappa) \leq 1$.

Remark 6.3. When $\text{Br}(\kappa)[p] \neq 0$, every class $[A] \in \text{Br}(K)[p]$ decomposes as $[A] = [\pi, \gamma]_{p,K} + [U]$ where $[U] \in \text{Br}(\kappa)[p]$ is the unramified component. The extension $K(\wp^{-1}(\pi))$ splits $[\pi, \gamma]$ but not $[U]$. If $[U_A] = [U_B] = 0$, Theorem 6.1 gives a common splitting field. Otherwise, let L/K be a common splitting field. Its maximal unramified subextension L_0/K has residue field extension $\bar{L}_0/\kappa$, and the unramified components $[U_A], [U_B]$ must be split by $\bar{L}_0$. Thus a necessary condition for EDCS over K is the existence of a common splitting field for $[U_A], [U_B]$ over κ . Whether this condition is also sufficient is not known.

7. FURTHER REMARKS AND OPEN QUESTIONS

The CDVF classifications of Sections 5 and 6 rely on the tame symbol exact sequence, which is special to Henselian discretely valued fields. For general fields, we indicate the structural difficulties.

7.1. Characteristic $\neq p$: an additive obstruction. Let $A = (a, b)_p$, $B = (c, d)_p$ with $\text{ind}(A \otimes B) = p$. The Matzri shortening technique [11] reduces the existence of a common subfield to the following: does there exist $s \in N_{F(\sqrt[p]{c})/F}(F(\sqrt[p]{c})^\times)$ such that

$$ac + s \cdot d \in N_{F(\sqrt[p]{a})/F}(F(\sqrt[p]{a})^\times) ?$$

This is a condition about a *sum* of elements of multiplicative subgroups landing in another multiplicative subgroup. Standard tools in Brauer group theory, Milnor K -theory, and Galois cohomology operate purely multiplicatively and do not address such additive–multiplicative mixing.

The index-drop condition $\text{ind}(A \otimes B) = p$ in $K_2(F)/p$ is $\{a, b\} + \{c, d\} = \{e, f\}$. The K_2 relation controls the product of reduced norm images, not their intersection. A simple group-theoretic illustration: in $(\mathbb{Z}/p\mathbb{Z})^2$, subgroups $U = \langle e_1 \rangle$, $V = \langle e_2 \rangle$ satisfy $UV = G$ but $U \cap V = \{0\}$.

7.2. Characteristic p : non-Henselian fields. For fields of characteristic p that are not CDVFs (e.g. rational function fields $\mathbb{F}_p(t)$), the EDCS problem remains open. The tame symbol exact sequence has no analogue for such fields. Explicit constructions of division algebras without common subfields exist in characteristic $\neq p$ (Jacob–Wadsworth [9]); a positive-characteristic analogue would require ASW symbols with sufficiently independent entries in $F/\wp(F)$.

7.3. Open questions.

- (1) For a general field F of characteristic $\neq p$, classify the common cyclic subfields of two degree- p division algebras A, B with $\text{ind}(A \otimes B) = p$.
- (2) For a CDVF $K = \kappa((\pi))$ of characteristic p with $\text{Br}(\kappa)[p] \neq 0$, determine whether the necessary condition in Remark 6.3 is also sufficient.

- (3) For a general field of characteristic p , determine whether a common Artin–Schreier splitting field always exists when $\text{ind}(A \otimes B) < p^2$.

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