ROST KERNEL OF DECOMPOSABLE DIVISION ALGEBRAS OVER COMPLETE DISCRETE VALUATION FIELDS

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Abstract. Let p be an odd prime, F a complete DVF of characteristic 0 with $\mu_p \subset F$, and $D \simeq (a_1, b_1)_F \otimes_F (a_2, b_2)_F$ a decomposable central division algebra of index p^2 and period p. We prove a rank barrier: $\operatorname{rank}(\Phi) = 2 \Rightarrow \operatorname{ind}(D) \leq p$, hence $\operatorname{ind}(D) = p^2 \Rightarrow \operatorname{rank}(\Phi) \geq 3$. We establish an inclusion chain $N \subseteq S$, $N \subseteq U^\perp$, $U^\perp \subseteq R$ with dimension formula $\dim N = d_F - 2 + k - t$ and $U^\perp = N \iff t - k = \operatorname{rank}(\Phi) - 2$ (assuming $\dim F^\times/F^{\times p} < \infty$). Over HDVF: $U^\perp = \{0\}$ unconditionally in mixed/ramified cases; in the unramified case with $H^3(K) = 0$, $\operatorname{Rost}(D)/F^{\times p} = H^1(K, \mu_p)$.

1. Introduction

Rost constructed his invariant before December 1992 [2, p. 98], as a generator of the cyclic group of degree 3 cohomological invariants of an algebraic group. [1, B.11] provided the first geometric realization. We are interested in Rost invariants of absolutely simple, simply connected groups of inner type A_n.

For a central division algebra D over a field F of characteristic 0 containing μ_p, the Rost invariant is the normalized map $r_{\mathrm{SL}_1(D)} \colon F^\times / \operatorname{Nrd}(D^\times) \to H^3(F, \mu_p^{\otimes 2})$, $a \operatorname{Nrd}(D^\times) \mapsto (a) \cup [D]$, defined in [2, 2.2] (see Section 2 for the precise definition). Its kernel $\operatorname{Rost}(D) = \{\lambda \in F^\times : (\lambda) \cup [D] = 0\}$ is the *Rost kernel*. The *Suslin group*, introduced by Merkurjev [7, §1], is $\operatorname{Suslin}(D) = \prod_{i | \operatorname{per} D} \operatorname{Nrd}((D^{\otimes i})^\times)^i$ (Definition 2.3); when $\operatorname{per}(D) = p$, $\operatorname{Suslin}(D) = \operatorname{Nrd}(D^\times) F^{\times p}$. The general goal is to determine the structure of $\ker(r_{\mathrm{SL}_1(D)})$. If G is a simply connected group, then $\ker(r_G) \simeq A^1(Y_\xi, K_2)$ by [2, p.128, 9.6]; we focus on $G = \mathrm{SL}_1(D)$.

When $\operatorname{char} F \nmid \operatorname{per} D$, the Rost kernel admits a geometric description $\operatorname{Rost}(D) = \operatorname{im}(t \colon H^1(\mathrm{SB}(D), K_2) \to F^\times)$ [7, p.323, after 1.5], and Merkurjev [7, §1, 1.6] proves

$$\operatorname{Suslin}(D) = \operatorname{im}\bigl(K_1(\mathrm{SB}(D))^{(1)} \to H^1(\mathrm{SB}(D), K_2) \xrightarrow{t} F^\times\bigr),$$

where $K_1(\mathrm{SB}(D))^{(1)}$ is the coniveau-1 K_1-subgroup. Therefore, the structure of $\ker(r_{\mathrm{SL}_1(D)}) = \operatorname{Rost}(D)/\operatorname{Nrd}(D^\times)$ depends on the filtration

$$\operatorname{Nrd}(D^\times) \subseteq \operatorname{Suslin}(D) \subseteq \operatorname{Rost}(D) \subseteq F^\times.$$

For the *generic* decomposable algebra $D \simeq (a_1, t_1)_F \otimes_F (a_2, t_2)_F$ over $F = k(t_1, t_2)$, Merkurjev [7, Prop. 2.5] computed $\operatorname{Rost}(D)/\operatorname{Suslin}(D) \cong \{a_1, a_2\}^\perp/N$. [5, Th.1.7] proved $\operatorname{Rost}(D) = \operatorname{Suslin}(D)$ over henselian DVF

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with residue cohomological dimension ≤ 2. For a non-generic algebra of decomposable type, these equalities require additional hypotheses. We prove that $R = U^{\perp}$ is false in the unramified HDVF case with $H^3(K) = 0$ (Remark 3.10); in the ramified HDVF case and over general fields with $\operatorname{rank}(\Phi) \geq 3$ it remains open.

Main results. Let p be an odd prime. Let F be a field of characteristic 0 containing a primitive p-th root of unity ζ_p. Let

$$(a,b)_F = F\langle x, y \mid x^p = a,\ y^p = b,\ yx = \zeta_p xy\rangle$$

denote a cyclic algebra of degree p for $a, b \in F^{\times}$.

For the classes $(a_1), (a_2) \in H^1(F, \mu_p)$, we write the orthogonal complement

$$\{a_1, a_2\}^{\perp} := \{\, v \in H^1(F,\mu_p) : v \cup (a_1) = v \cup (a_2) = 0 \,\},$$

with respect to the cup product $\cup\colon H^1(F,\mu_p) \times H^1(F,\mu_p) \to H^2(F,\mu_p)$.

Let $D \simeq (a_1,b_1)_F \otimes_F (a_2,b_2)_F$ with $\operatorname{ind}(D) = p^2$ and $\operatorname{per}(D) = p$. Set $E_1 := F(a_1^{1/p}), E_2 := F(a_2^{1/p}),\ E := F(a_1^{1/p}, a_2^{1/p})$. The Hochschild–Serre spectral sequence for E/F provides cohomological invariants

$$k := \dim\ker\big(H^2(\mathrm{Gal}(E/F),\mu_p) \xrightarrow{\inf} H^2(F,\mu_p)\big),$$

$$t := \dim \hat{H}^0\big(\mathrm{Gal}(E/F),\ H^1(E,\mu_p)\big),$$

satisfying $t \geq k$. We now state the main results.

Theorem 1.1 (Rank barrier)**.** *Let p be an odd prime, F a field of characteristic 0 containing μ_p, and $a_1, a_2 \in F^{\times}$ with $(a_1) \cup (a_2) \neq 0$ in $H^2(F,\mu_p)$. Define $\Phi(v) = (v \cup a_1, v \cup a_2)$. If $\operatorname{rank}(\Phi) = 2$, then any decomposable algebra $D \simeq (a_1,b_1)_F \otimes_F (a_2,b_2)_F$ satisfies $\operatorname{ind}(D) \leq p$, and consequently $\operatorname{ind}(D) = p^2 \Rightarrow \operatorname{rank}(\Phi) \geq 3$.*

Theorem 1.2 (Dimension formulas)**.** *Let p be an odd prime, F a field of characteristic 0 with $\mu_p \subset F$, and $D \simeq (a_1,b_1)_F \otimes_F (a_2,b_2)_F$ with $\operatorname{ind}(D) = p^2$, $\operatorname{per}(D) = p$, $(a_1) \cup (a_2) \neq 0$, and $\dim_{\mathbb{F}_p} F^{\times}/F^{\times p} < \infty$. Set $N := \operatorname{cor}(H^1(E,\mu_p))$, $U^{\perp} := \{a_1,a_2\}^{\perp}$, $r := \operatorname{rank}(\Phi)$, $R := \operatorname{Rost}(D)/F^{\times p}$, $S := \operatorname{Suslin}(D)/F^{\times p}$. Then $N \subseteq U^{\perp}$, $\dim N = d_F - 2 + k - t$, $\dim U^{\perp} = d_F - r$, $N = U^{\perp} \iff t - k = r - 2$, and $\operatorname{ind}(D) = p^2 \Rightarrow r \geq 3$.*

For henselian discrete valuation fields, the residue map deals with ramification of Brauer classes, but it had not been systematically applied to the quotient $\dfrac{\operatorname{Rost}(D)}{\operatorname{Suslin}(D)}$. [5, Th.1.7] proved $\operatorname{Rost}(D) = \operatorname{Suslin}(D)$ over henselian DVF with $\operatorname{cd}_p(K) \leq 2$. The next theorem relaxes this: we assume only $H^3(K, \mu_p^{\otimes 2}) = 0$ (a weaker hypothesis, see Remark 3.11) and compute $\operatorname{Rost}(D)/F^{\times p}$ unconditionally, with the full quotient $\operatorname{Rost}(D)/\operatorname{Suslin}(D)$ conditional on $S = N$.

Theorem 1.3. *Let p be an odd prime, F a complete DVF with residue field K ($\operatorname{char} K \neq p$), $\mu_p \subset F$, uniformizer t. Let $D \simeq (a_1,b_1)_F \otimes_F (a_2,b_2)_F$ with $\operatorname{ind}(D) = p^2$, $\operatorname{per}(D) = p$, and $(a_1) \cup (a_2) \neq 0$. Let $v\colon F^{\times}/F^{\times p} \to \mathbb{F}_p$ be the valuation mod p.*

(i) *(Mixed/ramified.) If $v(a_1) \neq 0$ or $v(a_2) \neq 0$, then $U^\perp = \{0\}$, $N = \{0\}$, and $\operatorname{rank}(\Phi) \geq 3$ (no finite-dimension hypothesis needed).*
(ii) *(Unramified.) If $v(a_1) = v(a_2) = 0$, $\partial([D]) = 0$, $H^3(K, \mu_p^{\otimes 2}) = 0$, and $S = N$, then $\operatorname{Rost}(D)/\operatorname{Suslin}(D) \cong H^1(K, \mu_p)/N_{E_0/K}(E_0^\times)K^{\times p}$. Quantitative dimension formulas for this quotient require $\dim_{\mathbb{F}_p} F^\times/F^{\times p} < \infty$ (Corollary 3.13).*

(Part (i) is Theorem 3.2; part (ii) follows from Theorems 3.9 and 3.12.)

In the unramified case, the computation of $\operatorname{Rost}(D)/F^{\times p}$ in Theorem 3.9 uses the residue isomorphism directly and does *not* require $R = U^\perp$. By contrast, the formula $\operatorname{Rost}(D)/\operatorname{Suslin}(D) \cong U^\perp/N$ would follow from $R = U^\perp$ and $S = N$; the former is known to be false in the unramified HDVF setting (Remark 3.10). The paper thus provides two independent routes to the Rost–Suslin quotient, one conditional on $R = U^\perp$ and the other unconditional (in the unramified HDVF case).

Structure of the paper. Section 2 fixes notation, proves the rank barrier and the Hochschild–Serre dimension formulas over arbitrary fields. Section 3 specialises to henselian DVF with $\operatorname{char} K \neq p$: ramified cases are settled unconditionally ($U^\perp = \{0\}$), while the unramified case is treated via the H^3 residue map. Section 4 analyses the lower obstruction in the unramified case. Section 5 summarises the classification and collects open problems.

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2. Preliminaries

2.1. **Kummer theory.** Let p be an odd prime and F a field of characteristic 0 containing a primitive p-th root of unity ζ_p. The Kummer isomorphism identifies

$$F^\times/F^{\times p} \xrightarrow{\sim} H^1(F, \mu_p), \qquad a \mapsto (a),$$

and the Merkurjev–Suslin theorem [8] identifies

$${}_p\operatorname{Br}(F) := \operatorname{Br}(F)[p] \xrightarrow{\sim} H^2(F, \mu_p).$$

The cup product

$$\cup \colon H^1(F, \mu_p) \times H^1(F, \mu_p) \to H^2(F, \mu_p)$$

is alternating. For a cyclic algebra $(a, b)_F$, $[(a, b)_F] = (a) \cup (b) \in H^2(F, \mu_p)$. Write $d_F := \dim_{\mathbb{F}_p} F^\times/F^{\times p}$ and $\operatorname{rad}_F := \{x \in F^\times/F^{\times p} : x \cup y = 0 \ \forall y\}$.

Remark 2.1. The dimension formulas in Sections 2–4 (involving d_F, d_K, and the Hochschild–Serre invariants) assume $d_F := \dim_{\mathbb{F}_p} F^\times/F^{\times p} < \infty$. The rank barrier (Theorem 1.1) and the vanishing $U^\perp = \{0\}$ (Lemma 3.1) do not require this hypothesis. The assumption $d_F < \infty$ holds, for instance, when F is a henselian DVF whose residue field K satisfies $\dim_{\mathbb{F}_p} K^\times/K^{\times p} < \infty$ (then $d_F = d_K + 1$).

2.2. **Rost invariant, Rost kernel, and Suslin group.**

Definition 2.1 (Rost invariant of $\mathrm{SL}_1(D)$)**.** Let F be a field of characteristic 0 containing μ_p. Let D be a central division algebra over F. The *Rost invariant* is the normalized map [2, 2.2]

$$r_{\mathrm{SL}_1(D)}\colon F^\times/\operatorname{Nrd}(D^\times) \longrightarrow H^3(F,\mu_p^{\otimes 2}), \qquad a\operatorname{Nrd}(D^\times) \longmapsto (a)\cup[D],$$

where $(a) \in H^1(F,\mu_p)$ via Kummer theory, $[D] \in H^2(F,\mu_p) \cong {}_p\operatorname{Br}(F)$ via Merkurjev–Suslin, and $\cup$ is the usual cup product. The invariant is normalized so that $\operatorname{im}(r_{\mathrm{SL}_1(D)}) \subseteq H^3(F,\mu_p^{\otimes 2})$ [2, 11.5]. (For the general definition over arbitrary characteristic, see [2, 2.2]; this paper works exclusively in characteristic 0.)

Definition 2.2 (Rost kernel)**.** after [7, p.326, 1.10]. Let F be a field of characteristic 0 containing μ_p. Let D be a central division algebra over F. Define the *Rost kernel* of D as

$$\operatorname{Rost}(D) := \{\lambda \in F^\times : (\lambda)\cup[D] = 0 \text{ in } H^3(F,\mu_p^{\otimes 2})\}.$$

When $\operatorname{char} F \nmid \operatorname{per} D$, the Rost invariant is the image of an injection

$$\operatorname{Rost}(D) = \operatorname{im}(t\colon H^1(\mathrm{SB}(D),K_2) \to F^\times)$$

where $H^1(\mathrm{SB}(D),K_2)$ is the first Zariski cohomology of the Milnor K_2 sheaf on the Severi–Brauer variety $\mathrm{SB}(D)$ by [7, p323, after 1.5].

Definition 2.3 (Suslin group)**.** after [7, p.324, Rem]. Let F be a field. Let D be a central division algebra over F. For each positive integer i, let $D^{\otimes i} = D\otimes_F \cdots \otimes_F D$ (i times). The *Suslin group* of D is the subgroup of $F^\times$.

$$\operatorname{Suslin}(D) := \prod_{i\mid \operatorname{per} D} \operatorname{Nrd}((D^{\otimes i})^\times)^i,$$

Specifically, $\operatorname{Suslin}(D)$ is the subgroup of $F^\times$ generated (as a multiplicative group) by all elements of the form $\operatorname{Nrd}(z)^i$ with $z \in (D^{\otimes i})^\times$ and $i \mid \operatorname{per} D$ (the product in Definition 2.3 is a product of *subgroups*, not a Cartesian product). The quotient $S := \operatorname{Suslin}(D)/F^{\times p}$ is well defined because $F^{\times p} \subseteq \operatorname{Suslin}(D)$ (Remark 2.2).

Remark 2.2. When $\operatorname{per}(D) = p$, the factor $i = p$ contributes $\operatorname{Nrd}((D^{\otimes p})^\times)^p$. Since $D^{\otimes p} \cong M_{p^2}(F)$ is split, its reduced norm is the determinant, which is surjective onto $F^\times$; hence $\operatorname{Nrd}((D^{\otimes p})^\times)^p = F^{\times p}$. Thus $F^{\times p} \subseteq \operatorname{Suslin}(D)$ for every central simple algebra of period p, and in fact $\operatorname{Suslin}(D) = \operatorname{Nrd}(D^\times)F^{\times p}$ (since the $i = 1$ factor contributes $\operatorname{Nrd}(D^\times)$ and the $i = p$ factor contributes $F^{\times p}$).

2.3. **Setup for the Decomposable Case.** Assume $D \cong (a_1,b_1)_F \otimes_F (a_2,b_2)_F$ with $\operatorname{ind}(D) = p^2$ and $\operatorname{per}(D) = p$. Set $E_i := F(a_i^{1/p})$ ($i = 1,2$) and $E := E_1E_2 = F(a_1^{1/p},a_2^{1/p})$. From $\operatorname{ind}(D) = p^2$ we have $\operatorname{Gal}(E/F) \cong C_p \times C_p$. We assume throughout

$$(a_1)\cup(a_2) \neq 0 \quad \text{in } H^2(F,\mu_p), \tag{1}$$

i.e. the cyclic algebra $(a_1, a_2)_F$ is not split. This does not follow from $\mathrm{ind}(D) = p^2$ alone.

Let $\mathrm{res} : H^1(F, \mu_p) \to H^1(E, \mu_p)$ and $\mathrm{cor} : H^1(E, \mu_p) \to H^1(F, \mu_p)$ be the restriction and corestriction. Under the Kummer isomorphisms $F^\times/F^{\times p} \cong H^1(F, \mu_p)$ and $E^\times/E^{\times p} \cong H^1(E, \mu_p)$, the restriction res corresponds to the natural inclusion and the corestriction cor corresponds to the norm $N_{E/F}$ modulo p-th powers [9, Prop. 3.4.10]. Since E/F is a finite extension, $d_E < \infty$ follows from $d_F < \infty$ (see Remark 2.1). The projection formula [9, Prop. 3.4.10] reads

$$\mathrm{cor}(\alpha) \cup \beta = \mathrm{cor}(\alpha \cup \mathrm{res}(\beta)), \qquad \alpha \in H^1(E, \mu_p),\ \beta \in H^1(F, \mu_p). \tag{2}$$

Merkurjev–Suslin [8] gives $\mathrm{cor}(H^1(E_i, \mu_p)) = \{v \in H^1(F, \mu_p) : v \cup (a_i) = 0\}$. Hence by Lemma 2.5,

(3)

$$\mathrm{cor}(H^1(E, \mu_p)) \subseteq \{v \in H^1(F, \mu_p) : v \cup (a_1) = v \cup (a_2) = 0\} =: \{a_1, a_2\}^\perp.$$

Lemma 2.3. *The following hold unconditionally (only* (1) *is required):* $\mathrm{res}|_{\{a_1,a_2\}^\perp}$ *is injective,* $U \cap U^\perp = \{0\}$*, and* $\mathrm{rank}(\Phi) \geq 2$*. If in addition* $\dim_{\mathbb{F}_p} H^1(F, \mu_p) < \infty$*, then* $\dim\{a_1, a_2\}^\perp = d_F - \mathrm{rank}(\Phi)$ *where* $\Phi\colon H^1(F, \mu_p) \to H^2(F, \mu_p)^2$, $\Phi(v) = (v \cup (a_1),\, v \cup (a_2))$.

Proof. For injectivity: if $v \in U^\perp := \{a_1, a_2\}^\perp$ and $\mathrm{res}(v) = 0$, then $v \in U \cap U^\perp$. Write $v = c_1a_1 + c_2a_2$ with $c_1, c_2 \in \mathbb{F}_p$. From $v \cup (a_1) = 0$ we get $c_2\,(a_1) \cup (a_2) = 0$, hence $c_2 = 0$ by (1); similarly $c_1 = 0$. Thus $v = 0$, proving $U \cap U^\perp = \{0\}$ and the injectivity of $\mathrm{res}|_{U^\perp}$.

For the dimension: $\dim U^\perp = d_F - \mathrm{rank}(\Phi)$ by definition. If $c_1\Phi_1 + c_2\Phi_2 = 0$ as functionals on $H^1(F, \mu_p)$, then for every v, $0 = c_1\Phi_1(v) + c_2\Phi_2(v) = c_1(v \cup a_1) + c_2(v \cup a_2) = v \cup (c_1a_1 + c_2a_2)$. Thus $c_1a_1 + c_2a_2 \in \mathrm{rad}_F$. Cupping with a_1 and a_2 as above forces $c_1 = c_2 = 0$; hence Φ_1, Φ_2 are linearly independent and $\mathrm{rank}(\Phi) \geq 2$. The exact value of $\mathrm{rank}(\Phi)$ depends on the field F: for sufficiently non-degenerate cup products one has $\mathrm{rank}(\Phi) = 2$, but over HDVFs in the unramified case the rank is 3 (see Remark 2.7). ☐

Proof of Theorem 1.1. Since $\mathrm{rank}(\Phi) = 2$ and $\Phi(a_1) = (0, h)$, $\Phi(a_2) = (-h, 0)$ where $h := (a_1) \cup (a_2) \neq 0$, the image of Φ is spanned by these two elements. By Lemma 2.3, $U \cap U^\perp = \{0\}$ and $\dim U = 2$. Since $\mathrm{rank}(\Phi) = 2$, the quotient $V/U^\perp \cong \mathrm{im}\,\Phi$ has dimension 2. The composition $U \hookrightarrow V \twoheadrightarrow V/U^\perp$ is injective (because $U \cap U^\perp = \{0\}$) between two vector spaces of the same finite dimension 2, hence an isomorphism. Therefore $V = U \oplus U^\perp$.

Write $b_i = u_i + w_i$ with $u_i \in U$, $w_i \in U^\perp$ $(i = 1, 2)$. By alternativity of the cup product and the definition of $U^\perp$, $a_1 \cup w_1 = -w_1 \cup a_1 = 0$, $a_2 \cup w_2 = -w_2 \cup a_2 = 0$, $a_1 \cup w_2 = -w_2 \cup a_1 = 0$, $a_2 \cup w_1 = -w_1 \cup a_2 = 0$. Therefore

$$[D] = (a_1) \cup (u_1 + w_1) + (a_2) \cup (u_2 + w_2) = (a_1) \cup u_1 + (a_2) \cup u_2.$$

Since $u_1, u_2 \in U = \langle a_1, a_2 \rangle$, each term $(a_i) \cup u_j$ is a multiple of $h = (a_1) \cup (a_2)$. Thus $[D] \in \langle h \rangle$.

The non-zero elements of $\langle h \rangle$ are represented by the degree-p cyclic algebra $(a_1, a_2)_F$ and its powers, all of which have index at most p. Hence $\mathrm{ind}(D) \leq p$. ☐

Remark 2.4. As noted in Theorem 1.1, $\mathrm{ind}(D) = p^2$ forces $\mathrm{rank}(\Phi) \geq 3$; the equality $R = U^\perp$ is open when $\mathrm{rank}(\Phi) \geq 3$ (Remark 2.8).

Remarks on the proof. (i) The proof uses only the bilinearity and alternating property of the cup product, which hold for the Milnor symbol in arbitrary characteristic [3, Chap. 7]. Hence the rank barrier is characteristic-independent. (ii) The proof does not require $\dim F^\times/F^{\times p} < \infty$ (only that $\mathrm{im}(\Phi)$ has finite dimension 2, which follows from $\mathrm{rank}(\Phi) = 2$).

Lemma 2.5 (Easy inclusion). *$N_{E/F}(E^\times)F^{\times p} \subseteq N_{E_1/F}(E_1^\times)F^{\times p} \cap N_{E_2/F}(E_2^\times)F^{\times p}$.*

Proof. $N_{E/F} = N_{E_i/F} \circ N_{E/E_i}$. □

2.4. **Structure of the Rost Kernel for the Decomposable Case.** Let $V := H^1(F, \mu_p) \cong F^\times/F^{\times p}$. Set $U := \langle a_1, a_2\rangle = \ker(\mathrm{res}\colon V \to H^1(E, \mu_p))$ and $U^\perp := \{a_1, a_2\}^\perp = \{v \in V : v \cup (a_1) = v \cup (a_2) = 0\}$. Define

$$R := \mathrm{Rost}(D)/F^{\times p}, \qquad S := \mathrm{Suslin}(D)/F^{\times p}, \qquad N := \mathrm{cor}(H^1(E, \mu_p)).$$

By Merkurjev–Suslin [8], $N_i := \mathrm{cor}(H^1(E_i, \mu_p)) = \{v \in V : v \cup (a_i) = 0\}$. By Lemma 2.5, $N \subseteq N_1 \cap N_2 = U^\perp$.

Lemma 2.6 (Basic inclusions). *For the decomposable case with* $\mathrm{ind}(D) = p^2$*,* $\mathrm{per}(D) = p$ *and* (1)*:*

(i) $N \subseteq S$ (the field E is a maximal subfield of D, hence $N_{E/F}(E^\times) \subseteq \mathrm{Nrd}(D^\times)$);
(ii) $N \subseteq U^\perp$ (the projection formula, as in (3)*);*
(iii) $U^\perp \subseteq R$ (if $v \cup (a_1) = v \cup (a_2) = 0$ then $v \cup [D] = v \cup (a_1) \cup (b_1) + v \cup (a_2) \cup (b_2) = 0$).

Proof. (i) The extension $E = F(a_1^{1/p}, a_2^{1/p})$ splits D (both a_1, a_2 become p-th powers in E, so each cyclic factor $(a_i, b_i)_F$ splits over E) and $[E : F] = p^2 = \mathrm{ind}(D)$. By the Köthe theorem (every splitting field of degree equal to the index embeds as a maximal subfield; see [10, Theorem 3.3.8] or [3, Theorem 7.5.11]), E is isomorphic to a maximal subfield of D. For x in that maximal subfield, $\mathrm{Nrd}_D(x) = N_{E/F}(x)$; hence $N_{E/F}(E^\times) \subseteq \mathrm{Nrd}(D^\times)$. Taking the quotient by $F^{\times p}$ gives $N \subseteq S$. (ii) For $x \in E^\times$, $\mathrm{cor}((x)) \cup (a_i) = \mathrm{cor}((x) \cup \mathrm{res}(a_i)) = \mathrm{cor}(0) = 0$, since $\mathrm{res}(a_i) = 0$ in $H^1(E, \mu_p)$. (iii) If $v \cup (a_1) = v \cup (a_2) = 0$, then $v \cup [D] = 0 \cup (b_1) + 0 \cup (b_2) = 0$, so $v \in R$. □

Remark 2.7. In the unramified HDVF case, the tame symbol gives each map $v \mapsto v \cup (a_i)$ an image of dimension up to 2 (one from $H^1(K)$, one from the ramified direction (t)). When the two images are independent, $\mathrm{rank}(\Phi) = 3$ and $\dim U^\perp = d_K - 2$; when they coincide, $\mathrm{rank}(\Phi) = 2$ and $\dim U^\perp = d_K - 1$. For q-adic local fields ($q \neq p$, $\mu_p \subset K$), $\dim H^1(K) = 2$ and a Zariski-open set of pairs (a_1, a_2) gives $\mathrm{rank}(\Phi) = 3$. Theorem 1.1 excludes $\mathrm{rank}(\Phi) = 2$ when $\mathrm{ind}(D) = p^2$.

Thus the subgroups of V obey

$$N \subseteq S, \qquad N \subseteq U^\perp, \qquad U^\perp \subseteq R, \tag{4}$$

while S and $U^\perp$ need not be comparable a priori.

The paper [7] computes $R = U^\perp$ and $S = N$ for the *generic* algebra $T = A_\xi(a_1, t_1) \otimes A_\xi(a_2, t_2)$ over $F = k(t_1, t_2)$ with indeterminates t_1, t_2, via the specialisation argument in [7, Prop. 2.5]. For a non-generic algebra D, these equalities require additional hypotheses. We isolate them explicitly.

Definition 2.4. The algebra D is called *Suslin-norm equalised* (with respect to E/F) if $\mathrm{Nrd}(D^\times)F^{\times p} = N_{E/F}(E^\times)F^{\times p}$, i.e. $S = N$.

Remark 2.8. The equality $R = U^\perp$ is a natural desideratum: together with Suslin-norm equalisation ($S = N$) it would give $\mathrm{Rost}(D)/\,\mathrm{Suslin}(D) \cong U^\perp/N$. The only known proof that $R = U^\perp$ requires $\mathrm{rank}(\Phi) = 2$ (one decomposes $V = U \oplus U^\perp$ and analyses $\Psi|_{\mathrm{im}\,\Phi}$ as in [7, Prop. 2.5]), but Theorem 1.1 shows this hypothesis is incompatible with $\mathrm{ind}(D) = p^2$. For $\mathrm{rank}(\Phi) \geq 3$, the equality $R = U^\perp$ is *false in general*: in the unramified HDVF case with $H^3(K) = 0$, Theorem 3.9 gives $R = H^1(K, \mu_p) \supsetneq U^\perp$ (Remark 3.10). In the ramified HDVF case the question remains open. In the unramified HDVF case, Theorem 3.9 circumvents this by computing R directly via the residue isomorphism, without using $R = U^\perp$.

Remark 2.9. Hypothesis (1) ($(a_1) \cup (a_2) \neq 0$) is used throughout: it forces $U \cap U^\perp = \{0\}$ (Lemma 2.3) and $\mathrm{rank}(\Phi) \geq 2$. The injectivity of $\mathrm{res}|_{U^\perp}$ (Lemma 2.3) is used in the Hochschild–Serre computation below to identify $\dim N$ with $\dim \mathrm{res}(N)$; this step is independent of the value of $\mathrm{rank}(\Phi)$. The direct-sum decomposition $V = U \oplus U^\perp$ requires the additional hypothesis $\mathrm{rank}(\Phi) = 2$, which Theorem 1.1 shows is incompatible with $\mathrm{ind}(D) = p^2$. The Hochschild–Serre dimension formula $\dim N = d_F - 2 + k - t$ does *not* require $V = U \oplus U^\perp$ and remains valid for all $\mathrm{rank}(\Phi) \geq 2$.

2.5. **Hochschild–Serre exact sequence.** For the Galois extension E/F, the 5-term Hochschild–Serre sequence (with coefficients μ_p, trivial action) reads [4]

$$\begin{aligned}(5)\quad 0 \to H^1(\mathrm{Gal}(E/F), \mu_p) \xrightarrow{\mathrm{inf}} H^1(F, \mu_p) \xrightarrow{\mathrm{res}} H^1(E, \mu_p)^{\mathrm{Gal}(E/F)} \\ \xrightarrow{d_2} H^2(\mathrm{Gal}(E/F), \mu_p) \xrightarrow{\mathrm{inf}} H^2(F, \mu_p).\end{aligned}$$

Here $H^1(\mathrm{Gal}(E/F), \mu_p) \cong \mathbb{F}_p^2$, generated by the Kummer classes of a_1, a_2. Define

$$(6)\qquad k := \dim \ker\big(H^2(\mathrm{Gal}(E/F), \mu_p) \xrightarrow{\mathrm{inf}} H^2(F, \mu_p)\big).$$

The cohomology ring $H^*(C_p \times C_p, \mathbb{F}_p) \cong \mathbb{F}_p[x_1, x_2] \otimes \Lambda(y_1, y_2)$ ($\deg y_i = 1$, $\deg x_i = 2$) gives $\dim H^2(\mathrm{Gal}(E/F), \mu_p) = 3$ with basis $\{x_1, x_2, y_1 y_2\}$. Under Kummer isomorphisms, $y_1 y_2 \mapsto (a_1) \cup (a_2) \neq 0$ by (1), so $y_1 y_2 \notin \ker(\mathrm{inf})$ and $k \in \{0, 1, 2\}$. From exactness,

$$(7)\qquad \dim H^1(E, \mu_p)^{\mathrm{Gal}(E/F)} = d_F - 2 + k.$$

Let $N_G : H^1(E, \mu_p)_{\mathrm{Gal}(E/F)} \to H^1(E, \mu_p)^{\mathrm{Gal}(E/F)}$ be the norm map on coinvariants, induced by $N = \sum_{g \in \mathrm{Gal}(E/F)} g$. Define

$$(8)\qquad t := \dim \mathrm{coker}(N_G) = \dim \hat{H}^0(\mathrm{Gal}(E/F),\ H^1(E, \mu_p)).$$

Then $\mathrm{res} \circ \mathrm{cor}$ equals the algebraic norm, which factors through the coinvariants as N_G; hence $N_G(H^1(E, \mu_p)_{\mathrm{Gal}(E/F)}) = \mathrm{res}(\mathrm{cor}(H^1(E, \mu_p)))$. By (3),

$\mathrm{cor}(H^1(E,\mu_p)) \subseteq \{a_1,a_2\}^\perp$; by Lemma 2.3, $\mathrm{res}|_{\{a_1,a_2\}^\perp}$ is injective. Hence $\dim N_G(H^1(E,\mu_p)_{\mathrm{Gal}(E/F)}) = \dim \mathrm{cor}(H^1(E,\mu_p))$, and from

$$H^1(E,\mu_p)_{\mathrm{Gal}(E/F)} \xrightarrow{N_G} H^1(E,\mu_p)^{\mathrm{Gal}(E/F)} \to \hat{H}^0(\mathrm{Gal}(E/F), H^1(E,\mu_p)) \to 0$$

we have

$$\dim \mathrm{cor}(H^1(E,\mu_p)) = \dim H^1(E,\mu_p)^{\mathrm{Gal}(E/F)} - t = d_F - 2 + k - t. \tag{9}$$

Proof of Theorem 1.2. In the setting of Lemma 2.6, let $N := \mathrm{cor}(H^1(E,\mu_p))$, $U^\perp := \{a_1,a_2\}^\perp$, $r := \mathrm{rank}(\Phi)$, and k,t as in (6)–(8). From (9) we have $\dim N = d_F - 2 + k - t$, and from Lemma 2.3 $\dim U^\perp = d_F - r$. Hence

$$N = U^\perp \iff t - k = r - 2, \qquad t \geq k + (r-2).$$

If moreover $R = U^\perp$ (open in the ramified case when $r \geq 3$; Remark 2.8) and $S = N$ (Definition 2.4), then $\mathrm{Rost}(D)/\mathrm{Suslin}(D) \cong U^\perp/N$ and $\mathrm{Rost}(D) = \mathrm{Suslin}(D) \iff N = U^\perp \iff t - k = r - 2$. By Theorem 1.1, $r \geq 3$ when $\mathrm{ind}(D) = p^2$, so $t - k = r - 2 \geq 1$ and $t > k$. $\square$

Remark 2.10. For the generic algebra Merkurjev [7, Prop. 2.5] proves $R = U^\perp$ and $S = N$. For non-generic algebras the unconditional content is the inclusion chain (Lemma 2.6) and $\dim N = d_F - 2 + k - t$ (Theorem 1.2); $R = U^\perp$ and $S = N$ are not known in general (Remark 2.8).

Corollary 2.11. *From* $\dim N \leq \dim U^\perp$ *(since* $N \subseteq U^\perp$*) and* $\dim N = d_F - 2 + k - t$, $\dim U^\perp = d_F - r$, *we obtain* $t \geq k + (r-2)$. *When* $\mathrm{ind}(D) = p^2$, *Theorem 1.1 gives* $r \geq 3$, *hence* $t \geq k+1$.

3. The Decomposable Case over Henselian Discrete Valuation Fields

3.1. **Tame symbol and classification.** Let F be a henselian DVF with residue field K, $\mathrm{char}(K) \neq p$, $\mu_p \subset K$, and uniformizer t (for instance, the reader may take $F = K((t))$). The tame symbol exact sequence [3, Cor. 6.4.4] is stated for complete DVF; it extends to henselian DVF because $\mathrm{Br}(F) \cong \mathrm{Br}(\widehat{F})$ where $\widehat{F}$ is the completion [3, Prop. 6.4.5], and the residue field K is the same for F and $\widehat{F}$.

Warning on H^3 residues. The H^2 tame symbol extends to the henselian case via completion, but the residue map ∂_3 on $H^3(F, \mu_p^{\otimes 2})$ used in Theorems 3.9 and 3.12 requires F to be *complete*. The mixed/ramified results (Lemma 3.1, Theorem 3.2) use only the H^2 tame symbol and are valid over any henselian DVF. The tame symbol for F is defined via the completion $\widehat{F}$:

$$\partial\colon H^2(F,\mu_p) \xrightarrow{\sim} H^2(\widehat{F},\mu_p) \xrightarrow{\partial} H^1(K,\mu_p).$$

Explicitly,

$$\partial\big((x) \cup (y)\big) = (-1)^{v(x)v(y)}\, \overline{x^{v(y)} y^{-v(x)}}.$$

For odd p the sign $(-1)^{v(x)v(y)} = \pm 1$ lies in $K^{\times p}$ whenever $v(x)v(y) \equiv 0 \pmod p$; in the intersection computation of Lemma 3.1 the final condition forces $v(x) \equiv 0 \pmod p$, hence the sign is trivial. We retain the sign in the formula for completeness. The exact sequence

$$0 \to H^2(K,\mu_p) \to H^2(F,\mu_p) \xrightarrow{\partial} H^1(K,\mu_p) \tag{10}$$

is then valid for the henselian DVF F. The valuation exact sequence is

$$0 \to H^1(K,\mu_p) \to H^1(F,\mu_p) \xrightarrow{v \bmod p} \mathbb{F}_p \to 0, \tag{11}$$

so $d_F = d_K + 1$ where $d_K := \dim_{\mathbb{F}_p} H^1(K,\mu_p)$.

Let a_1, a_2 be the Kummer generators of the decomposable case. We classify them by valuation:

- **Mixed**: $v(a_1) = 1$, $v(a_2) = 0$.
- **Ramified**: $v(a_1) = v(a_2) = 1$, $\bar{a}_1 \neq \bar{a}_2$ in $H^1(K,\mu_p)$. (If $\bar{a}_1 = \bar{a}_2$, then $a_1/a_2 \in F^{\times p}$ after adjusting by a unit, forcing $(a_1,a_2)_F$ to split, contradicting (1).)
- **Unramified**: $v(a_1) = v(a_2) = 0$ ($a_1, a_2 \in K^\times$).

3.2. Ramified cases.

Lemma 3.1. *In the mixed and ramified cases over a henselian DVF,* $\{a_1,a_2\}^\perp = \{0\}$.

Proof. By Merkurjev–Suslin [8], $N_i := \mathrm{cor}(H^1(E_i,\mu_p)) = \{v \in V : v \cup (a_i) = 0\}$. Hence $U^\perp = N_1 \cap N_2$. For $v \in N_i$, the condition $v \cup (a_i) = 0$ in $H^2(F,\mu_p)$ implies $\partial(v \cup (a_i)) = 0$; hence $N_i \subseteq \ker \partial((a_i) \cup -)$. Using the tame symbol formula with sign $\partial((a) \cup (b)) = (-1)^{v(a)v(b)}\overline{a^{v(b)}b^{-v(a)}}$ [3, Cor. 6.4.4]:

- *Mixed case* ($a_1 = tu$, $a_2 = c \in K^\times \setminus K^{\times p}$): for $x = t^k\bar{x}$, $\partial((a_1) \cup x) = (-1)^k \bar{u}^k \bar{x}^{-1}$, $\partial((a_2) \cup x) = \bar{c}^k$. From $\partial((a_2) \cup x) = 0$ we obtain $\bar{c}^k = 1$, hence $k \equiv 0 \pmod p$. Replacing x by its valuation-0 representative $\bar{x}$ (since $t^k \in F^{\times p}$ when $p \mid k$), the sign becomes $(-1)^0 = 1$ and $\partial((a_1) \cup \bar{x}) = \bar{x}^{-1}$. Thus $\bar{x} = 1$ in $K^\times/K^{\times p}$, and the intersection is $\{0\}$.
- *Ramified case* ($a_1 = tu$, $a_2 = tv$, $\bar{u} \neq \bar{v}$): $\partial((a_1) \cup x) = (-1)^k \bar{u}^k \bar{x}^{-1}$, $\partial((a_2) \cup x) = (-1)^k \bar{v}^k \bar{x}^{-1}$. Setting both to 1 gives $(-1)^k \bar{u}^k = \bar{x} = (-1)^k \bar{v}^k$; the sign $(-1)^k$ cancels, yielding $\bar{u}^k = \bar{v}^k$. If $k \not\equiv 0 \pmod p$, then $\bar{u}^k = \bar{v}^k$ implies $(\bar{u}/\bar{v})^k = 1$; since $\partial((a_1) \cup (a_2)) = -\bar{u}/\bar{v} \neq 1$ (Merkurjev–Suslin) and $\bar{u}/\bar{v}$ has order p, this forces $k \equiv 0 \pmod p$, contradiction. Hence $k \equiv 0 \pmod p$, and replacing x by $\bar{x}$ as above gives $\bar{x} = 1$. Thus the intersection is $\{0\}$.

Therefore $N_1 \cap N_2 \subseteq \{0\}$, giving $U^\perp = \{0\}$. ☐

Theorem 3.2. *In the mixed and ramified cases over a henselian DVF,* $U^\perp = \{0\}$*,* $N = \{0\}$*, and* $\mathrm{rank}(\Phi) \geq 3$.

Remark 3.3. Since $U^\perp \subseteq R$, we have $\{0\} \subseteq R$; the reverse inclusion $R \subseteq U^\perp$ (hence $\mathrm{Rost}(D) = \mathrm{Suslin}(D)$) is open for $d_K \geq 2$ (Remark 2.8).

Remark 3.4. Lemma 3.1 shows $U^\perp = \{0\}$ in the mixed and ramified cases. When $d_F < \infty$, Lemma 2.3 gives $\dim U^\perp = d_F - \mathrm{rank}(\Phi)$, hence $d_F = \mathrm{rank}(\Phi)$; together with $d_F = d_K + 1$ we obtain $\mathrm{rank}(\Phi) = d_K + 1$. Thus $\mathrm{rank}(\Phi)$ is determined by the residue field dimension d_K: $d_K = 1$ gives $\mathrm{rank}(\Phi) = 2$, while $d_K \geq 2$ forces $\mathrm{rank}(\Phi) \geq 3$. **However**, by Theorem 1.1, $\mathrm{rank}(\Phi) = 2$ forces $\mathrm{ind}(D) \leq p$ for any decomposable algebra; therefore the case $d_K = 1$ cannot occur when $\mathrm{ind}(D) = p^2$. In the setting of this paper we must have $d_K \geq 2$ and $\mathrm{rank}(\Phi) \geq 3$.

Proposition 3.5. *In the mixed and ramified cases with $d_K \geq 2$ (the case $d_K = 1$ being excluded by Theorem 1.1), we have $U^\perp = \{0\}$ and $N = \{0\}$ unconditionally. The Suslin-norm equalisation condition $S = N$ is equivalent to $S = \{0\}$, i.e.* $\mathrm{Suslin}(D) = F^{\times p}$. *Whether this holds for specific residue fields K with $d_K \geq 2$ is not resolved here. For q-adic local fields ($q \neq p$) one has* $\mathrm{Nrd}(D^\times) = F^\times$ *for any central division algebra D by Hasse–Schilling, hence* $\mathrm{Suslin}(D) = F^\times$*; however such fields do not support algebras of index p^2 and period p (Remark 3.7).*

Proposition 3.6. *Assume $F = K((t))$ (as in §3.3). In the unramified case with $b_1, b_2 \in K^\times$ (which is automatic after adjusting by $F^{\times p}$, as argued before Theorem 3.9), the Suslin-norm equalisation condition for D over F is equivalent to the corresponding condition for the residue algebra $\overline{D} = (a_1, b_1)_K \otimes_K (a_2, b_2)_K$ over K:*

$$\mathrm{Nrd}_{\overline{D}}(\overline{D}^\times)K^{\times p} = N_{E_0/K}(E_0^\times)K^{\times p},$$

i.e. the B2 condition for $\overline{D}$ over K. ***However****, verifying that either side holds for a given residue field K is an open problem in general (Remark 3.7). Thus the reduction provided by this proposition does not by itself resolve the Suslin-norm equalisation question.*

Proof. Since D is inertial (unramified), $D \cong \overline{D} \otimes_K F$ as F-algebras. For $d \in \overline{D}^\times \subset D^\times$, the reduced norm satisfies $\mathrm{Nrd}_D(d) = \mathrm{Nrd}_{\overline{D}}(d)$ (inflated from K). Conversely, every $x \in D^\times$ can be written as $x = t^r \cdot d \cdot (1+z)$ where t is a uniformizer of F, $r = v_D(x)$, $d \in \overline{D}^\times$, and $z \in \mathfrak{m}_D$ (since D is inertial, $v_D(D^\times) = \mathbb{Z}$ and the residue of $t^{-r}x$ lies in $\overline{D}^\times$). By [11, Corollary 11.19], $\mathrm{Nrd}_D(1 + \mathfrak{m}_D) = 1 + \mathfrak{m}_F$. Since $\mathrm{char}\, K \neq p$, Hensel's lemma applied to $X^p - (1 + m)$ ($m \in \mathfrak{m}_F$) gives $1 + \mathfrak{m}_F \subseteq F^{\times p}$; hence the contribution of $1+\mathfrak{m}_F$ disappears modulo $F^{\times p}$. Moreover, for any $x \in D^\times$, $v_F(\mathrm{Nrd}_D(x)) = \mathrm{ind}(D) \cdot v_D(x) \in p^2\mathbb{Z}$ by [11, Proposition 1.5(i)], so $\mathrm{Nrd}_D(D^\times)$ contains no element of valuation p modulo p-th powers. Hence, modulo $F^{\times p}$, the uniformiser t contributes nothing and $\mathrm{Suslin}(D)/F^{\times p} = \mathrm{Nrd}_{\overline{D}}(\overline{D}^\times)K^{\times p}/K^{\times p}$ (as subspaces of $H^1(K, \mu_p) \subset H^1(F, \mu_p)$), while $\mathrm{cor}(H^1(E, \mu_p))$ intersected with $H^1(K)$ equals $N_{E_0/K}(E_0^\times)K^{\times p}/K^{\times p}$ by Theorem 3.12. The equality $S = N$ over F is therefore exactly the B2 condition over K. □

Remark 3.7. For the unramified decomposable case to be non-vacuous, the residue field K must satisfy simultaneously: (i) $H^3(K, \mu_p^{\otimes 2}) = 0$ (which holds when $\mathrm{cd}_p(K) \leq 2$); (ii) existence of a decomposable division algebra of index p^2 and period p over K. Number fields and q-adic local fields satisfy (i) but not (ii), since per = ind for all Brauer classes over these fields. Function fields of transcendence degree 2 over algebraically closed fields (e.g. $K = \mathbb{C}(x, y)$) satisfy $H^3(K) = 0$ and support the required algebras (for $p = 2$, the biquaternion algebra $(x, y) \otimes (x+1, xy)$ over $\mathbb{C}(x, y)$ has index 4 and period 2; for odd p, see [7, Cor. 2.9]). Theorem 3.9 applies to this example unconditionally (it does not require $\dim K^\times/K^{\times p} < \infty$). The dimension formulas (Theorem 1.2, Corollaries 2.11, 3.13) require $d_F < \infty$ and are not applicable to this example. We also note a necessary condition for the finite-dimensional case: if $\dim K^\times/K^{\times p} \leq 3$ then every element of ${}_p\mathrm{Br}(K) \cong$

$K_2^M(K)/p \cong \Lambda^2(K^\times/K^{\times p})$ is a symbol (for $d \leq 2$, Λ^2 has dimension ≤ 1; for $d = 3$, all bivectors are decomposable). Hence any algebra over K has index $\leq p$, contradicting (ii). Therefore any candidate residue field must satisfy $\dim K^\times/K^{\times p} \geq 4$.

Notwithstanding the conditional nature of the finite-dimensional hypotheses, the structural analysis of Theorem 3.12 has independent value: it identifies exactly which cohomological invariants control the Rost–Suslin quotient (the B2 quotient, the space $H^1(K)/U^\perp$, and Suslin-norm equalisation), and the dimension formula $t - k + 1$ expresses their relative sizes in terms of the Hochschild–Serre invariants of E/F.

3.3. **Unramified case.** In this subsection we assume $F = K((t))$ (or more generally, F contains a coefficient field lifting K, so that $K^\times$ embeds into $F^\times$ and every unramified division algebra over F is of the form $\overline{D} \otimes_K F$). This ensures that the inertial decomposition $D \cong \overline{D} \otimes_K F$ and the embedding $H^1(K, \mu_p) \hookrightarrow H^1(F, \mu_p)$ are available.

Now $a_1, a_2 \in K^\times$ with $(a_1, a_2)_K$ not split. Set $E_0 := K(a_1^{1/p}, a_2^{1/p})$, so $E = E_0((t))$.

Lemma 3.8. *For $a_i \in K^\times \setminus K^{\times p}$ and $\lambda \in H^1(F, \mu_p)$, if $(a_i) \cup \lambda = 0$ then $v(\lambda) \equiv 0 \pmod p$. Modifying by $F^{\times p}$, we may assume $\lambda \in H^1(K, \mu_p)$, and $(a_i) \cup \lambda = 0$ in $H^2(F, \mu_p)$ iff $(a_i) \cup \lambda = 0$ in $H^2(K, \mu_p)$.*

Proof. The tame symbol $\partial((a_i) \cup \lambda) = \bar{a}_i^{v(\lambda)}$ [3, Cor. 6.4.4]. If $(a_i) \cup \lambda = 0$, its tame symbol vanishes, so $\bar{a}_i^{v(\lambda)} = 1$. Since $\bar{a}_i$ has order p, $v(\lambda) \equiv 0 \pmod p$. Multiplying by $t^p \in F^{\times p}$, we may assume $v(\lambda) = 0$, i.e. $\lambda \in H^1(K, \mu_p)$. Then $(a_i, \lambda)_F$ is the unramified cyclic algebra inflated from K, and its class vanishes in $H^2(F, \mu_p)$ iff it vanishes in $H^2(K, \mu_p)$ [3, §6.4]. □

In the unramified case, we assume in addition that the Brauer class $[D]$ is unramified, i.e. $\partial([D]) = 0$ in $H^1(K, \mu_p)$. (This is a non-trivial restriction: $v(a_1) = v(a_2) = 0$ alone does not guarantee that D is unramified, as b_1 or b_2 may be ramified.) When $\partial([D]) = 0$, the formula $\partial([D]) = v(b_1)\bar{a}_1 + v(b_2)\bar{a}_2$ holds. Since $v(a_1) = v(a_2) = 0$, the class $(a_1) \cup (a_2)$ inflates from $H^2(K, \mu_p)$; by (1) it is non-zero, so $\bar{a}_1, \bar{a}_2$ are linearly independent in $H^1(K, \mu_p)$ (otherwise their cup product would vanish by alternativity). Hence $\partial([D]) = 0$ forces $v(b_1) \equiv v(b_2) \equiv 0 \pmod p$. Hence, after adjusting b_1, b_2 by $F^{\times p}$, we may assume $b_1, b_2 \in K^\times$. Thus $[D] \in H^2(K, \mu_p)$ is inflated from K.

For the next two theorems we assume $F = K((t))$ (a complete DVF with coefficient field K). Completeness is required for the residue map ∂_3 on H^3; the coefficient field ensures the embedding $K^\times \hookrightarrow F^\times$ and the inertial decomposition $D \cong \overline{D} \otimes_K F$ for unramified algebras.

Theorem 3.9. *Let F be a complete DVF with residue field K ($\mathrm{char}(K) \neq p$, $\mu_p \subset K$, $H^3(K, \mu_p^{\otimes 2}) = 0$). Let $D \simeq (a_1, b_1)_F \otimes_F (a_2, b_2)_F$ satisfy $v(a_1) = v(a_2) = 0$, $\partial([D]) = 0$, $\mathrm{ind}(D) = p^2$, $\mathrm{per}(D) = p$, and $(a_1) \cup (a_2) \neq 0$. Then*

$$\mathrm{Rost}(D)/F^{\times p} = H^1(K, \mu_p).$$

This theorem does not *require* $\dim F^\times/F^{\times p} < \infty$.

Proof. As argued above, the hypothesis $\operatorname{ind}(D) = p^2$ and $(a_1) \cup (a_2) \neq 0$ force $b_1, b_2 \in K^\times$ after adjusting by $F^{\times p}$; consequently $[D] \in H^2(K, \mu_p) \subset H^2(F, \mu_p)$ is inflated from K.

Let $\lambda \in H^1(F, \mu_p)$ and write $\lambda = t^k \bar{\lambda}$ with $k = v(\lambda) \bmod p$ and $\bar{\lambda} \in H^1(K, \mu_p)$ (using the decomposition $H^1(F, \mu_p) \cong H^1(K, \mu_p) \oplus \mathbb{F}_p$). Then

$$\lambda \cup [D] = k \cdot (t) \cup [D] + (\bar{\lambda}) \cup [D] \quad \text{in } H^3(F, \mu_p^{\otimes 2}),$$

where (t) denotes the Kummer class of the uniformiser. Since $[D]$ is inflated from K, the residue map $\partial_3 \colon H^3(F, \mu_p^{\otimes 2}) \to H^2(K, \mu_p)$ fits into the exact sequence $0 \to H^3(K, \mu_p^{\otimes 2}) \to H^3(F, \mu_p^{\otimes 2}) \xrightarrow{\partial_3} H^2(K, \mu_p)$ [3, Cor. 6.8.3] and satisfies $\partial_3(\pi \cup \alpha) = \alpha_K$ for any uniformizer π and any class $\alpha \in H^2(F, \mu_p)$ inflated from K, where $\alpha_K \in H^2(K, \mu_p)$ denotes the residue class [3, Prop. 6.8.4]. Taking $\pi = (t)$ (the Kummer class of the uniformiser) and $\alpha = [D]$ gives $\partial_3((t) \cup [D]) = [D]_K$, the class of $[D]$ in $\operatorname{Br}(K)[p]$; and $\partial_3((\bar{\lambda}) \cup [D]) = 0$ (because $(\bar{\lambda}) \cup [D]$ is inflated from $H^3(K, \mu_p^{\otimes 2})$, which vanishes by hypothesis). Hence

$$\partial_3(\lambda \cup [D]) = k \cdot [D]_K \quad \text{in } H^2(K, \mu_p).$$

Since $H^3(K, \mu_p^{\otimes 2}) = 0$, the residue ∂_3 is injective $H^3(F, \mu_p^{\otimes 2}) \hookrightarrow H^2(K, \mu_p)$. Thus $\lambda \cup [D] = 0$ in $H^3(F)$ if and only if $k \cdot [D]_K = 0$ in $H^2(K)$. Because $\operatorname{ind}(D) = p^2$, the class $[D]_K \in \operatorname{Br}(K)[p]$ is non-zero, so $k \cdot [D]_K = 0$ forces $k \equiv 0 \pmod{p}$.

Therefore $\operatorname{Rost}(D)/F^{\times p} = \{\lambda \in H^1(F, \mu_p) : v(\lambda) \equiv 0\} = H^1(K, \mu_p)$. $\square$

Remark 3.10. Theorem 3.9 shows that in the unramified case with $H^3(K, \mu_p^{\otimes 2}) = 0$, $\operatorname{Rost}(D)/F^{\times p} = H^1(K, \mu_p)$. By Lemma 3.8, $U^\perp \subseteq H^1(K, \mu_p)$. The hypothesis $(a_1) \cup (a_2) \neq 0$ implies $a_1 \notin U^\perp$ (since $a_1 \cup a_2 \neq 0$) while $a_1 \in H^1(K, \mu_p) = R$; hence $R \supsetneq U^\perp$ strictly, without any finite-dimension hypothesis. The unramified HDVF case with $H^3(K) = 0$ thus provides a *counterexample* to the equality $R = U^\perp$: the "open problem" of Remark 2.8 is therefore false in this regime.

Remark 3.11. The hypothesis $H^3(K, \mu_p^{\otimes 2}) = 0$ is *weaker* than $\operatorname{cd}_p(K) \leq 2$: the latter implies the former, but the converse requires control of all twisted p-torsion sheaves. We use $H^3(K) = 0$ throughout; the condition $\operatorname{cd}_p(K) \leq 2$ is mentioned only for readers familiar with the cohomological dimension formulation. **However**, an additional constraint applies in the unramified decomposable case: the residue field K must support a division algebra of index p^2 and period p (since D inflates from such an algebra over K). Classical fields with $\operatorname{cd}_p \leq 2$, such as number fields (Poitou–Tate) and q-adic local fields ($q \neq p$), satisfy per = ind for all Brauer classes (Albert–Brauer–Hasse–Noether theorem / local class field theory), and therefore admit no such algebra. Function fields of transcendence degree 2 over algebraically closed fields (e.g. $K = \mathbb{C}(x, y)$) satisfy $H^3(K) = 0$ and admit the required algebras; **however**, $\dim K^\times/K^{\times p}$ is infinite for such fields, so the dimension formulas of this paper do not apply (see Remarks 2.1 and 3.7).

Theorem 3.12. *In the unramified case, define the B2 quotient*

$$Q_{B2} := \frac{\{v \in H^1(K,\mu_p) : v \cup (a_1) = v \cup (a_2) = 0\}}{N_{E_0/K}(E_0^\times)K^{\times p}/K^{\times p}} \cong U^\perp/N.$$

If $H^3(K,\mu_p^{\otimes 2}) = 0$ *and* $S = N$*, then*

$$\frac{\mathrm{Rost}(D)}{\mathrm{Suslin}(D)} \cong \frac{H^1(K,\mu_p)}{N} \cong Q_{B2} \times H^1(K,\mu_p)/U^\perp \quad \textit{(non-canonical).}$$

Proof. By Lemma 3.8, any $\lambda \in U^\perp = \{a_1, a_2\}^\perp$ satisfies $v(\lambda) \equiv 0 \pmod p$, so after adjusting by $F^{\times p}$ we have $\lambda \in H^1(K,\mu_p)$, and $(a_i) \cup \lambda = 0$ in $H^2(F,\mu_p)$ iff $(a_i) \cup \lambda = 0$ in $H^2(K,\mu_p)$. Hence $U^\perp \subseteq H^1(K,\mu_p)$ and the numerator $\{v \in H^1(K,\mu_p) : v \cup (a_1) = v \cup (a_2) = 0\}$ equals $U^\perp$.

For the denominator: since E/F is unramified, $x = t^r x_0 u \in E^\times$ ($x_0 \in E_0^\times$, $u \in 1 + \mathfrak{m}_E$) has norm $N_{E/F}(x) \equiv t^{p^2 r} N_{E_0/K}(x_0) \pmod{F^{\times p}}$ by [11, Prop. 1.5] (1-units lie in $E^{\times p}$ by Hensel's lemma). Since $t^{p^2 r} \in F^{\times p}$, the image of $N = \mathrm{cor}(H^1(E,\mu_p))$ intersected with $H^1(K,\mu_p)$ is $N_{E_0/K}(E_0^\times)K^{\times p}/K^{\times p}$.

Now $Q_{\mathrm{B2}} = U^\perp/(N \cap H^1(K))$. Theorem 3.9 gives $\mathrm{Rost}(D)/F^{\times p} = H^1(K,\mu_p)$ (without using Remark 2.8 or the rank$(\Phi) = 2$ hypothesis). If D is Suslin-norm equalised ($S = N$), then

$$\mathrm{Rost}(D)/\,\mathrm{Suslin}(D) \cong H^1(K,\mu_p)/N \cong Q_{\mathrm{B2}} \times H^1(K,\mu_p)/U^\perp \quad \text{(non-canonical).}$$

Since $N \subseteq H^1(K,\mu_p)$ (all norms from an unramified extension are unramified), $N \cap H^1(K) = N$; in the finite-dimensional setting $\dim H^1(K)/N = \dim Q_{\mathrm{B2}} + \dim H^1(K)/U^\perp$, giving the (non-canonical) product decomposition. (Note: Remark 2.8 is not used here; the equality $R = H^1(K)$ comes from the residue isomorphism in Theorem 3.9, which does not require rank$(\Phi) = 2$.) In general, Lemma 2.6 gives $N \subseteq S$ and $U^\perp \subseteq R$, hence the image of $U^\perp$ in R/S is $U^\perp/(U^\perp \cap S)$. Since $N \subseteq U^\perp \cap S$, $\dim U^\perp/(U^\perp \cap S) \leq \dim Q_{\mathrm{B2}}$; thus $\dim Q_{\mathrm{B2}}$ is an *upper* bound for the $U^\perp$-contribution to R/S, not a lower bound. ☐

Corollary 3.13. ***If*** $H^3(K,\mu_p^{\otimes 2}) = 0$*,* $\partial([D]) = 0$*,* ***and*** D *is Suslin-norm equalised (Definition 2.4), then*

$$\dim \frac{\mathrm{Rost}(D)}{\mathrm{Suslin}(D)} = t - k + 1,$$

where k and t are the Hochschild–Serre invariants of the extension E/F defined in (6)–(8)*.* ***Warning:*** *the hypotheses include the existence of a residue field K with* $\dim K^\times/K^{\times p} < \infty$*,* $H^3(K) = 0$*, and supporting a decomposable division algebra of index p^2 and period p; the existence of such fields (or the construction of concrete examples) remains to be investigated (Remark 3.7). The corollary is therefore conditional and may be vacuous for the finite-dimensional case; Theorem 3.12 remains unconditional.*

Proof. By the finite-dimensionality hypothesis (Remark 2.1), $d_K < \infty$ and $d_F = d_K + 1 < \infty$. Theorem 3.9 gives $\mathrm{Rost}(D)/F^{\times p} = H^1(K,\mu_p)$, of dimension d_K. Formula (9) gives $\dim N = d_F - 2 + k - t = d_K - 1 + k - t$. Under Suslin-norm equalisation ($S = N$), $\mathrm{Rost}(D)/\,\mathrm{Suslin}(D) \cong H^1(K)/N$, of dimension $d_K - (d_K - 1 + k - t) = t - k + 1$. ☐

4. The Lower Obstruction in the Unramified Case

The lower obstruction is the quotient $F^{\times p}/(F^{\times p} \cap \mathrm{Nrd}(D^\times))$, which together with the upper obstruction $\mathrm{Rost}(D)/\,\mathrm{Suslin}(D)$ determines the full Rost kernel via the exact sequence

$$1 \to \frac{F^{\times p}}{F^{\times p} \cap \mathrm{Nrd}(D^\times)} \to \ker(r_{\mathrm{SL}_1(D)}) \to \frac{\mathrm{Rost}(D)}{\mathrm{Suslin}(D)} \to 1.$$

We analyse it using the Jacob–Wadsworth decomposition.

4.1. **General Structure over HDVFs.** Let F be a henselian DVF with residue field K, $\mathrm{char}(K) \neq p$, uniformiser t. For any tame central division F-algebra D, the Jacob–Wadsworth theorem [6, Theorem 5.15] provides a decomposition in the Brauer group

$$[D] = [I] + [N] \quad \text{in } \mathrm{Br}(F),$$

where I is an inertial (unramified) division algebra and N is a nicely semiramified (NSR) division algebra, with $\mathrm{ind}(D) = \mathrm{ind}(I)\cdot\mathrm{ind}(N)$ and $\mathrm{per}(D) = \mathrm{lcm}(\mathrm{per}(I), \mathrm{per}(N))$.

The reduced norm of D can be analysed through this decomposition using the valuation theory of Tignol–Wadsworth [11]. Recall the factorisation

$$F^{\times p} = t^{p\mathbb{Z}} \cdot K^{\times p} \cdot (1 + \mathfrak{m}_F)^p,$$

where $(1+\mathfrak{m}_F)^p \subseteq \mathrm{Nrd}_D(D^\times)$ by [11, Corollary 11.19] (the reduced norm of the principal unit group is the full principal unit group).

Remark 4.1. The lower obstruction for the ramified decomposable case remains open. In the unramified case (Theorem 4.2), the valuation argument succeeds because D is inertial: every reduced norm has valuation divisible by p^2, forcing $t^p \notin \mathrm{Nrd}(D^\times)$. In the ramified case, the Jacob–Wadsworth decomposition $D \cong I \otimes_F N$ mixes an inertial factor I and a nicely semiramified factor N; the reduced norm of a general element is not a pure tensor, and the interaction between the two factors via non-pure-tensor terms obstructs a straightforward filtration argument. Resolving this would require a finer analysis of the graded reduced norm or a restriction to residue fields where Hasse–Schilling surjectivity holds (e.g. q-adic local fields).

Theorem 4.2. *In the unramified decomposable case over an HDVF with $\partial([D]) = 0$ and $b_1, b_2 \in K^\times$, the lower obstruction satisfies $\dim_{\mathbb{F}_p}\big(F^{\times p}/(F^{\times p} \cap \mathrm{Nrd}(D^\times))\big) \geq 1$, with the class of t^p generating a subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z}$.*

Proof. Since D is inertial, every $d \in D^\times$ satisfies $v_D(d) \in \mathbb{Z}$. By [11, Proposition 1.5(i)], $v_F(\mathrm{Nrd}_D(d)) = \mathrm{ind}(D) \cdot v_D(d) = p^2 \cdot v_D(d) \in p^2\mathbb{Z}$. Hence $\mathrm{Nrd}_D(D^\times) \subseteq \{x \in F^\times : v_F(x) \equiv 0 \pmod{p^2}\}$. The element t^p has valuation $p \notin p^2\mathbb{Z}$ (as $p < p^2$), so $t^p \notin \mathrm{Nrd}_D(D^\times)$ while $t^p \in F^{\times p}$. Thus t^p is non-trivial of order p in $F^{\times p}/(F^{\times p} \cap \mathrm{Nrd}(D^\times))$. □

Remark 4.3. The remaining contribution to the lower obstruction in the unramified case comes from the residue field factor $K^{\times p}/(K^{\times p} \cap \mathrm{Nrd}_{\overline{D}}(\overline{D}^\times))$. This is the *lower obstruction of the residue algebra* $\overline{D}$ over K, which is a division algebra of decomposable type of index p^2 and period p over K. Its analysis depends on K and is not pursued here.

5. Conclusion and open problems

For any central division F-algebra D of period p, the Rost kernel fits into

$$1 \to \underbrace{\frac{F^{\times p}}{F^{\times p} \cap \mathrm{Nrd}(D^\times)}}_{\text{lower}} \to \ker(r_{\mathrm{SL}_1(D)}) \to \underbrace{\frac{\mathrm{Rost}(D)}{\mathrm{Suslin}(D)}}_{\text{upper}} \to 1.$$

- The **upper obstruction**: Lemma 3.1 gives $U^\perp = \{0\}$ unconditionally in mixed/ramified configurations, but $\mathrm{Rost}(D) = \mathrm{Suslin}(D)$ requires $R = U^\perp$ and $S = N$, which remain open for $d_K \geq 2$ (Theorem 3.2). Theorem 1.1 excludes the case $d_K = 1$. Unramified with $H^3(K) = 0$ gives $\mathrm{Rost}(D)/F^{\times p} = H^1(K)$ (Theorem 3.9) and the B2 quotient Q_{B2} (Theorem 3.12).
- The **Suslin-norm equalisation**: open in ramified cases (Proposition 3.5); for unramified cases, Proposition 3.6 reduces $S = N$ over F to the corresponding condition for the residue algebra over K, whose verification is open (Remark 3.7).
- The **lower obstruction** ($F^{\times p} \subseteq \mathrm{Nrd}(D^\times)$): open for the decomposable case over HDVF in general.

5.1. Classification table.

	ramified†	**unramified**‡
HDVF type	$d_K \geq 2$	$H^3(K) = 0$
$U^\perp$**,** N	$= \{0\}$ (uncond.)	$U^\perp \subseteq H^1(K)$, $N = N_{E_0/K}(E_0^\times)K^{\times p}$
$R/F^{\times p}$	open	$= H^1(K, \mu_p)$ (Thm 3.9)
Upper (R/S**)**	open (requires R and S)	$Q_{\mathrm{B2}} \oplus (\mathbb{Z}/p)^{\mathrm{rank}(\Phi)-1}$ **if** $S = N$ (Thm 3.12); else open
S-N eq.	open (Prop 3.5: $S = \{0\}$)	open (Rem 3.7)
Lower	open (Rem 4.1)	$\geq \mathbb{Z}/p\mathbb{Z}$ (Thm 4.2)
ker	open	open

†Mixed and ramified cases (includes mixed configurations with one ramified and one unramified generator). The case $d_K = 1$ (which would imply $\mathrm{rank}(\Phi) = 2$) is excluded by Theorem 1.1: it forces $\mathrm{ind}(D) \leq p$, incompatible with $\mathrm{ind}(D) = p^2$. For $d_K \geq 2$, $U^\perp = \{0\}$ and $N = \{0\}$ unconditionally (Lemma 3.1); the equality $R = U^\perp$ is open (Remark 2.8).

‡The unramified case requires a residue field K with $H^3(K) = 0$ that supports a decomposable division algebra of index p^2 and period p. Number fields and q-adic local fields are excluded (per = ind). Valid examples include function fields of surfaces over algebraically closed fields (Remark 3.7).

To summarise: in the ramified configuration, $U^\perp = \{0\}$ and $N = \{0\}$ unconditionally, so the upper obstruction $\mathrm{Rost}(D)/\mathrm{Suslin}(D)$ reduces to R/S with R unknown and $S = \{0\}$ conditionally on Suslin-norm equalisation. In the unramified configuration with $H^3(K) = 0$, $\mathrm{Rost}(D)/F^{\times p} = H^1(K, \mu_p)$ unconditionally; the B2 quotient and the dimension formula $t - k + 1$ for $\mathrm{Rost}(D)/\mathrm{Suslin}(D)$ are conditional on $S = N$. The central open problems are: (i) $R = U^\perp$ in ramified HDVF configurations with $\mathrm{rank}(\Phi) \geq 3$; (ii) Suslin-norm equalisation $S = N$ in all cases; (iii) the lower obstruction in ramified configurations; (iv) explicit residue fields satisfying the hypotheses of the unramified case.

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