

# AI MATHEMATICIAN AS A PARTNER IN ADVANCING MATHEMATICAL DISCOVERY

**Anonymous authors**

Paper under double-blind review

## ABSTRACT

Artificial intelligence (AI) has demonstrated impressive progress in mathematical reasoning, yet its integration into the practice of mathematical research remains limited. In this study, we investigate how the AI Mathematician (AIM) system can operate as a research partner rather than a mere problem solver. Focusing on a challenging problem in homogenization theory, we analyze the autonomous reasoning trajectories of AIM and incorporate targeted human interventions to structure the discovery process. Through iterative decomposition of the problem into tractable subgoals, selection of appropriate analytical methods, and validation of intermediate results, we reveal how human intuition and machine computation can complement one another. This collaborative paradigm enhances the reliability, transparency, and interpretability of the resulting proofs, while retaining human oversight for formal rigor and correctness. The approach leads to a complete and verifiable proof, and more broadly, demonstrates how systematic human-AI co-reasoning can advance the frontier of mathematical discovery.

## 1 INTRODUCTION

In recent years, artificial intelligence (AI) has made remarkable progress in mathematical reasoning, achieving milestones once thought to be exclusive to human intelligence. In mathematical competitions, large language models (LLMs) have demonstrated outstanding performance. For example, several LLMs have achieved scores exceeding 90 on the AIME benchmarks (GPT, 2025; o4-, 2025; QwenTeam, 2025; Gro, 2025), which are constructed from real American Invitational Mathematics Examination (AIME) problems (AIM, 2025a;b), and some have even reached perfect 100-point scores (GPT, 2025; QwenTeam, 2025; Gro, 2025). Furthermore, Gemini with Deep Think has officially attained a gold-medal standard at the 66th International Mathematical Olympiad (IMO 2025) (Luong & Lockhart, 2025), marking a symbolic moment in the competitive mathematical performance of AI. As competition-based benchmarks begin to approach saturation, Glazer et al. (2025) introduced FrontierMath, a challenging new evaluation suite composed of problems crafted by expert mathematicians that typically require hours of deliberate reasoning for experts to solve. Notably, the o4-mini-medium model (o4-, 2025) has been reported to outperform the average human team on this benchmark (Ho, 2025), underscoring the growing ability of AI to engage with complex, research-level mathematics. Beyond competition-style problem solving, progress has also emerged in AI-assisted mathematical discovery. Romera-Paredes et al. (2024) and Novikov et al. (2025) demonstrate that LLMs can facilitate genuine mathematical discovery through guided program search. Similarly, GPT-5-Thinking (GPT, 2025) has been credited with helping renowned researchers resolve a challenging quantum computing problem (Aronson & Witteveen, 2025). Taken together, these developments suggest that AI is beginning to move beyond solving predefined problems toward a more engaged role in mathematical exploration.

Despite substantial progress, a considerable gap remains between current AI capabilities and the requirements of genuine mathematical research. In competitive settings, such as mathematical contests, problems are typically solved within minutes or hours. Even demanding benchmarks like FrontierMath challenge expert mathematicians for only several hours. By contrast, authentic mathematical research often unfolds over

049 months or even years of sustained reasoning, conceptual innovation, and proof development. For instance,  
050 it took more than seven years for Andrew Wiles to complete the proof of Fermat’s Last Theorem (Singh,  
051 1997), illustrating the temporal and intellectual depth of true mathematical inquiry. In the pursuit of  
052 mathematical discovery, existing representative AI systems are grounded in search-based paradigms that  
053 face inherent scalability and generalization constraints. Approaches such as FunSearch (Romera-Paredes  
054 et al., 2024) and AlphaEvolve (Novikov et al., 2025) depend on problems being formalizable in pro-  
055 grammatic form, thereby limiting their applicability to only a subset of mathematical domains. Similarly,  
056 AlphaGeometry (Trinh et al., 2024) and its successor AlphaGeometry2 (Chervonyi et al., 2025) focus  
057 exclusively on geometric reasoning, leaving vast areas of mathematics unexplored. While there are cases  
058 in which AI systems have contributed valuable insights—such as the work by Aaronson & Witteveen  
059 (2025)—the completion and validation of proofs ultimately remain the domain of human researchers.  
060 This highlights the current limitations of AI in conducting autonomous, creative, and deeply integrative  
061 mathematical research.

062 To mitigate these limitations, this study explores the potential of more collaborative and interactive AI  
063 systems capable of engaging in sustained reasoning and iterative refinement. We investigate the application  
064 of AIM (Liu et al., 2025), a multi-agent framework developed for autonomous mathematical exploration  
065 and proof generation, as a research partner in advancing mathematical inquiry. AIM harnesses the  
066 capabilities of LLMs to iteratively formulate conjectures, verify proofs, optimize reasoning pathways and  
067 refine proof details. By incorporating human expertise through a through several representative interaction  
068 paradigm, the framework seeks to integrate the complementary strengths of human intuition and machine  
069 intelligence in addressing complex, research-level mathematical problems.

070 In particular, we examine the use of AIM in tackling a challenge in homogenization theory, a disci-  
071 pline concerned with characterizing the macroscopic behavior of heterogeneous materials by averaging  
072 their microscopic properties across multiple scales. Through systematic analysis of AIM’s intermediate  
073 reasoning trajectories and targeted human interventions, we decompose the overarching problem into  
074 tractable subgoals, guide the selection of appropriate analytical methodologies, and rigorously validate  
075 the correctness of intermediate results. This human-AI co-reasoning paradigm improves the reliability,  
076 transparency, and interpretability of candidate proofs and produces auditable reasoning traces, while  
077 recognizing that formal rigor and final correctness require human oversight. Through this collaborative  
078 approach, we successfully derive a complete proof (Appendix C) for the aforementioned problem and,  
079 in the process, systematize representative modes of human-AI interaction while distilling key empirical  
080 insights. We believe this study provides meaningful guidance for AI-assisted mathematical research and  
081 establishes a foundation for deeper human-AI collaboration in advancing mathematical discovery.

082 In summary, our main contributions are as follows:

- 083 • We advocate a human-AI collaborative paradigm for mathematical research that integrates the  
084 computational capabilities of AI systems with the intuition and domain expertise of human  
085 mathematicians.
- 086 • We conduct a case study on a challenging problem in homogenization theory, resulting in a  
087 rigorous proof spanning nearly nineteen pages (Appendix C). A substantial portion of this proof  
088 is generated by AI, which makes nontrivial contributions throughout the process, demonstrating  
089 the potential of the paradigm for tackling complex, research-level mathematical problems.
- 090 • We systematize modes of human-AI interaction and extract empirical insights that can inform  
091 the design of future AI-assisted mathematical research frameworks, which may also serve as a  
092 practical guideline for mathematicians seeking to leverage AI in their own research.

## 093 2 PRELIMINARIES

### 094 2.1 THE HOMOGENIZATION PROBLEM

095 The mathematical research problem we investigate in this work is an instance of a Stokes–Lamé transmis-  
096 sion system with a vanishing fluid inclusion, analyzed in the homogenization regime  $\varepsilon \rightarrow 0$ . This problem  
097

will be referred to as the **Homogenization Problem** in the rest of this work. The complete description of the problem is provided in Appendix A.

Consider the displacement field  $u_\varepsilon$  satisfying the following coupled **Lamé-Stokes system**:

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u_\varepsilon = 0 & \text{in } \Omega \setminus \bar{D}_\varepsilon, \\ \mathcal{L}_{\tilde{\mu}}(u_\varepsilon, p_\varepsilon) = 0 \text{ and } \operatorname{div} u_\varepsilon = 0 & \text{in } D_\varepsilon, \\ u_\varepsilon|_- = u_\varepsilon|_+ \text{ and } \frac{\partial(u_\varepsilon, p_\varepsilon)}{\partial\nu(\infty, \tilde{\mu})}\Big|_- = \frac{\partial u_\varepsilon}{\partial\nu(\lambda, \mu)}\Big|_+ & \text{on } \partial D_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial\nu(\lambda, \mu)}\Big|_{\partial\Omega} = g \in H_{\mathbb{R}}^{-\frac{1}{2}}(\partial\Omega) \text{ and } u_\varepsilon|_{\partial\Omega} \in H_{\mathbb{R}}^{\frac{1}{2}}(\partial\Omega). \end{cases} \quad (1)$$

Here,  $\mathcal{L}_{\lambda,\mu} u_\varepsilon = \mu \Delta u_\varepsilon + (\lambda + \mu) \nabla \operatorname{div} u_\varepsilon$  and  $\mathcal{L}_{\tilde{\mu}}(u_\varepsilon, p_\varepsilon) = \tilde{\mu} \Delta u_\varepsilon + \nabla p_\varepsilon$ .

We need to conclude the limit homogenization equation as the scale of the cell tends to be zero  $\varepsilon \rightarrow 0$ . At the same time, we wonder the estimate between the original solution  $u_\varepsilon$  and the limited solution  $u_{\text{lim}}$ , i.e.,

$$\|u_\varepsilon - u_{\text{lim}}\|_{H^1(\Omega)} \lesssim \varepsilon^\alpha \quad (2)$$

for some  $\alpha \in (0, 1)$ . It is necessary to analyze and obtain determined value of  $\alpha$ , and strictly prove this conclusion.

## 2.2 AIM: AN AI MATHEMATICIAN SYSTEM

AIM is a multi-agent framework built upon large language models (LLMs) for conducting mathematical research (Liu et al., 2025). Its design addresses two fundamental challenges: the intrinsic complexity of mathematical theory and the rigor of reasoning processes. On one hand, AIM incorporates an *exploration and memory mechanism* that decomposes complex problems into multi-step explorations, generates intermediate conjectures, and iteratively reuses verified lemmas to refine reasoning. On the other hand, AIM employs *Pessimistic Rational Verification (PRV)*, in which multiple independent verifiers evaluate each intermediate proof, and a proof is judged as incorrect if any verifier deems it incorrect.

AIM consists of three core agents—the explorer, verifier, and optimizer—along with a memory module. The explorer is responsible for generating conjectures and constructing candidate proofs. The verifier independently examines the logical correctness of each proof step, while the optimizer refines and corrects errors identified during verification. The memory module archives automatically verified lemmas and other intermediate results for future reference. The agents operate in an iterative manner: given a problem, the explorer first proposes a sequence of intermediate conjectures accompanied by detailed proofs, potentially leveraging information stored in the memory module. The verifier evaluates their validity, and if errors are detected, the optimizer revises the proofs based on feedback from the verifier.

AIM has achieved satisfactory results in some theories (Liu et al., 2025). However, it still had numerous limitations and errors regarding the homogenization problem. Nevertheless, AIM made notable progress on this problem, and its intermediate results and proof sketches were found by human researchers to be highly insightful. This naturally leads to a key question: **Can human expertise be harnessed to guide AIM toward completing the proof for the homogenization problem?** This question forms the basis of our investigation into human-AI collaboration in mathematical research.

## 3 OVERVIEW

Based on an AI-human collaborative paradigm, we successfully completed the proof of the homogenization problem. The full proof is presented in Appendix C. Throughout this process, **we deliberately minimized human intervention**, allowing AIM to autonomously explore, reason, and construct mathematical arguments as much as possible. This design enables us to better understand both the strengths and limitations of AIM in mathematical research. The main conclusion for the homogenization problem is summarized as follows: We derived the homogenization equation in limit case and denoted its solution as  $u_{\text{lim}}$  (Eq. 41 in

147 Appendix C). The error estimation between the limit solution  $u_{\text{lim}}$  and the original solution  $u_\varepsilon$  under the  
 148 scale  $\varepsilon$  is further explored. We have analyzed that the  $\alpha = \frac{1}{2}$  and strictly proven the following conclusion:  
 149

$$150 \quad \|u_\varepsilon - u_{\text{lim}}\|_{H^1(\Omega)} \lesssim \varepsilon^{\frac{1}{2}}. \quad (3)$$

152 Building on the autonomous reasoning results of AIM, we observed that the original problem exceeds the  
 153 current capability of AIM; however, AIM could successfully tackle reduced or simplified subproblems and  
 154 showed strong ability to recognize and apply relevant mathematical theories even without explicit guidance.  
 155 This observation brings three key advantages: (1) it reduces the need for detailed human supervision  
 156 in solving subproblems; (2) it promotes exploration of diverse solution paths; and (3) it accelerates the  
 157 overall proof process. Building on these insights, we divided the original homogenization problem into  
 158 six subproblems and guided AIM through each. By weaving together the resulting proofs, we ultimately  
 159 arrived at a complete and coherent solution to the problem.  
 160

161 The six subproblems include: (1) *Two-Scale Expansion*, (2) *Cell Problem and Homogenization Equation*,  
 162 (3) *Existence and Uniqueness*, (4) *Ellipticity of Operator*, (5) *Error Estimation and Control*, and (6)  
 163 *Regularity of Cell Problem*. The objectives of each subproblem and the respective roles of AIM and human  
 164 experts are summarized below.

- 165 • *Two-Scale Expansion*: By performing a two-scale expansion on the equation, we obtain equations  
 166 corresponding to different scales. AIM made various errors in complex symbolic reasoning tasks.  
 167 Here, we have manually derived the content of this work.  
 168
- 169 • *Cell Problem and Homogenization Equation*: Based on the equations at different scales, we derive  
 170 the homogenization equation and construct the cell problem. AIM has insufficient understanding  
 171 of this task so this task has been completed manually.  
 172
- 173 • *Existence and Uniqueness*: AIM explored and applied reasonable theorems to analyze the operator  
 174 properties of the equation, and the proof process was obtained after we manually adjusted and  
 175 refined the details.  
 176
- 177 • *Ellipticity of Operator*: This subproblem concerns the ellipticity of the operator, which is a  
 178 fundamental property. AIM provides a proof with a relatively high degree of completion.  
 179
- 180 • *Error Estimation and Control*: This is the most complex subproblem, requiring a rigorous  
 181 analytical process and detailed derivation process. Based on the output results of AIM, we  
 182 manually adjusted and decomposed the subproblem, and finally obtained the complete proof  
 183 process by manually refining results of AIM.  
 184
- 185 • *Regularity of Cell Problem*: This is the conclusion put forward by AIM during its autonomous  
 186 exploration. After multiple human-AI interaction experiments and with appropriate theoretical  
 187 guidance, AIM finally produced a complete and correct proof.

188 The most representative AI-human interaction occurred during the resolution of the *Error Estimation and*  
 189 *Control* subproblem, which is the most complex one. Upon closer examination of AIM’s results, we found  
 190 a certain property of cell problem without providing a proof for it. This prompted deeper analysis, revealing  
 191 that establishing this property required a more profound understanding of the underlying theory than  
 192 initially assumed. In other words, *this process helped the human expert better grasp the intrinsic difficulty of*  
 193 *the problem*. The human expert then suggested that AIM consider mathematical tools such as the Difference  
 194 Quotient (LeVeque, 2007), Galerkin Method (Galerkin, 1915), and Schauder Theory (Ladyzhenskaya  
 195 et al., 1968; Boccia, 2013; Zhuge, 2021), providing only their definitions without procedural hints or proof  
 outlines. Eventually, AIM succeeded in proving the property using Schauder Theory. This experience  
 vividly illustrates that ***although AIM may still be a flawed individual researcher today, it can already***  
***serve as a valuable research partner—if used wisely.***

## 4 MODES OF HUMAN-AI INTERACTION

In pursuing the complete proof of the homogenization problem, we found that effective human-AI collaboration plays a crucial role. Based on extensive experimentation, we summarize five representative modes of interaction that proved particularly effective:

- **Direct Prompting** (Sec. 4.1). This mode guides the agent toward promising proof directions and optimizes its reasoning path through targeted yet concise instructions. It can be further divided into three subtypes: *Theorem Prompts* (Sec. 4.1.1), *Conceptual Guidance* (Sec. 4.1.2), and *Detail Refinement* (Sec. 4.1.3).
- **Theory-Coordinated Application** (Sec. 4.2). In this mode, the agent is provided with a coherent body of mathematical theory, enabling it to derive related results within the theoretical framework. Unlike Theorem Prompts, which focus on specific goals, this mode emphasizes the integration and application of an entire theoretical system.
- **Interactive Iterative Refinement** (Sec. 4.3). This mode follows a “Feedback – Revision – Re-reasoning” cycle, through which human experts and AIM collaboratively refine and complete proofs, leading to a more coherent and rigorous reasoning process.
- **Applicability Boundary and Exclusion Domain** (Sec. 4.4). Certain tasks—such as decomposing proof strategies or constructing problem formulations—remain challenging for AIM. We therefore recommend assigning these tasks to human experts, while reserving AIM’s involvement for domains where it demonstrates reliability and insight.
- **Auxiliary Optimization Strategies** (Sec. 4.5). These strategies enhance the correctness and robustness of proofs by iteratively providing additional contextual information and optimizing the selection or combination of mathematical tools.

### 4.1 DIRECT PROMPTING

Direct Prompting is an approach that provides clear human guidance, directing AIM’s attention to critical elements in the proof process, thereby minimizing irrelevant or off-topic reasoning to the greatest extent. This approach is particularly effective during the initial construction phase of the proof framework, or when handling critical steps that require conceptual breakthroughs.

- **Applicable Conditions:** The solution approach is foreseeable by the human, and the required theoretical components are known, but the derivation details are tedious. We can constrain AIM to use only the given knowledge, thereby reducing autonomous conjecture.
- **Typical Mathematical Scenarios:** For relatively simple subproblems, we can quickly identify the applicable theorem or method and directly prompt AIM to use the corresponding tool. When validating conclusions or conjectures, we can provide guidance to let AIM explore the process and record intermediate findings.
- **Expected Outcomes:** AIM correctly applies the provided theorems and completes the proof within the given knowledge pack, with preconditions checked and steps properly cited.

The prompts can be categorized into three common and distinct types:

- **Theorem Prompts:** Directly supplying formal statements, assumptions, or target theorems to anchor AIM’s reasoning process.
- **Conceptual Guidance:** Offering high-level descriptions of strategies, intuitions, or mathematical insights that contextualize the problem.
- **Detail Refinement:** Providing step-specific instructions or local corrections to ensure logical coherence and technical precision at finer granularity.

#### 4.1.1 THEOREM PROMPTS

During experiments, we directly supplied AIM with theorems and lemmas relevant to the current stage of the proof, thus anchoring its reasoning process within a well-defined mathematical foundation. This form of guidance proves to be highly effective in narrowing AIM’s focus and structuring the proof pathway.

For instance, in the proof of the cell problem, we explicitly introduced auxiliary lemmas from Schauder theory and provided them as prompts to AIM. Under this prompt, the output of AIM indicates that it proactively and appropriately incorporates the prompt information and executes the correct derivational procedure.

##### Prompt to AIM

###### *Disclaimer*

*For clarity, we format prompts using typographic styles (e.g., bold text) and render formulas as symbols rather than raw LaTeX code. “Content” is one of the labels used by AIM to distinguish different components and can be safely ignored when interpreting the prompt. The detailed prompt and response are in appendix B. The same applies below.*

###### **Content:**

You can use the following theorem to analyze the problem, integrating the content of the theorem and its derivation process.

###### *Disclaimer*

*The outputs of AIM include content such as `\begin{lemma}\end{lemma}`. We do not modify these outputs. LaTeX automatically assigns the numbering. Please ignore the numbers and the same applies below.*

**Lemma 1.** *Some prompts are omitted here for clarity.*

where  $\chi, r$  are weak solutions. Moreover, for all  $\rho \in (0, t)$ , integrating the above inequalities yields:

$$\left( \int_{B(\rho)} |\chi - l|^2 \right)^{\frac{1}{2}} + \rho |g_0 - \zeta(l, q)| \leq \left| \frac{\rho}{t} \right|^{\beta+1} \left( \left( \int_{B(t)} |\chi|^2 \right)^{\frac{1}{2}} + t |g_0 - \zeta(l, q)| \right).$$

###### *Note*

*This derivation process yields a crucial conclusion and embodies the inherent logic of the inferential steps*

##### Response of AIM

###### *Disclaimer*

*For clarity, we render formulas as symbols rather than raw LaTeX code. The same applies below.*

**\*\*Step 2: Bounding  $\int_{B(t)} |\chi - l|^2 dx$ \*\*** From Lemma 4 and scaling arguments (see hint-5), we derive:

$$\int_{B(t)} |\chi - l|^2 dx \leq C_2 t^{2(\beta+1)} \int_{B(t)} |\nabla \chi|^2 dx.$$

#### 4.1.2 CONCEPTUAL GUIDANCE

When analyzing the proof strategy for a given problem, AIM often exhibits vague or circular reasoning. In such cases, we—as human experts—can provide high-level framework prompts drawn from relevant mathematical fields. This allows the agents to conduct proof experiments within a conceptually sound structure, which can then be refined through iterative analysis of its outputs.

294 In the early experiment, we observed that AIM proposed an estimate form of the error:

$$295 \quad u_\varepsilon^{\text{APP}}(x) = u_0(x) + \varepsilon \chi^{D_x u_0(x)}\left(\frac{x}{\varepsilon}\right) \quad \text{in } \Omega_\varepsilon. \quad (4)$$

297 Upon careful analysis of the experimental results, it became evident that the proposed form was inaccurate,  
298 and AIM’s derivation process lacked sufficient and accurate detail.

### 299 Response of AIM without guidance

301 By the coercivity and the conjecture:

$$302 \quad c\|\mathbf{e}\|_{H^1(\Omega)}^2 \leq a(\mathbf{e}, \mathbf{e}) = |\text{Res}(\mathbf{e})| \leq C\varepsilon^{\frac{1}{2}}\|\mathbf{u}_0\|_{H^1(\Omega)}\|\mathbf{e}\|_{H^1(\Omega)}.$$

303 *Some responses are omitted here for clarity.*

304 Combining:

$$305 \quad \|\mathbf{u}_0 - \mathbf{u}_\varepsilon\|_{H^1(\Omega)} \leq \left(\frac{C}{c}C'\right)\varepsilon^{\frac{1}{2}}\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}.$$

306 Therefore, we revisited the problem from the perspective of homogenization error and, in subsequent proof  
307 attempts, provided a mathematically sound formulation to serve as the estimated error form and use it as  
308 guiding information to prompt AIM. And AIM finally offered a more reasonable proof process.

### 309 Prompt to AIM

310 **Content:** Consider the following form of the error.

311 *Some prompts are omitted here for clarity.*

312 Suppose

$$313 \quad w_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\eta_\varepsilon S_\varepsilon^2(\nabla u_0).$$

314 *Note*

315 *Here, we present the error form of the homogenization problem, constructed via convolution.*

### 316 Response of AIM with guidance

317 Since  $D \subset \Omega \setminus \Omega_{2\varepsilon}$ , the standard mollifier-estimate (Lemma 5) yields

$$318 \quad \|S_\varepsilon^2(\nabla u_0) - \nabla u_0\|_{L^2(D)} \leq \|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})},$$

319 and clearly  $\|\nabla\psi\|_{L^2(D)} \leq \|\nabla\psi\|_{L^2(\Omega)}$ . Hence

$$320 \quad |J_D| \leq C\|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})}\|\nabla\psi\|_{L^2(\Omega)}.$$

## 321 4.1.3 DETAIL REFINEMENT

322 For reasoning details generated by AIM—such as formula derivations, symbol definitions, and computa-  
323 tional steps—directly identifying deviations and providing corrective guidance is an effective strategy for  
324 local error calibration. Similarly, proof accuracy can be enhanced by prompting targeted revisions within  
325 specific segments of the argument.

### 326 Prompt to AIM

327 **Content:** *Some prompts are omitted here for clarity.*

328 where  $\mathbf{u} = (u^1, \dots, u^d)$  represents the displacement field and the divergence of  $\mathbf{u}$  is given by

$$329 \quad \text{div } \mathbf{u} = \sum_{i=1}^d \frac{\partial u^i}{\partial x_i}.$$

**Note**

The operational rules are emphasized here through specific formulas.

## 4.2 THEORY-COORDINATED APPLICATION

The Theory-Coordinated Application paradigm is an interaction mode that constrains the agents to operate within a pre-established theoretical framework to conduct derivations or proofs. It is intended for relatively complex intermediate problems where the solution path is foreseeable but detailed reasoning is nontrivial.

- **Application Conditions:** This paradigm is appropriate when the problem is relatively complex and demands a multi-step chain of reasoning. The human can encode the theory’s prior knowledge that the model will follow the inherent derivation logic.
- **Typical Mathematical Scenarios:** This paradigm targets scenarios where the final theorem or proposition can be obtained by progressively applying the selected theory’s intrinsic logic, moving stepwise from definitions and lemmas to the desired conclusion.
- **Expected Outcomes:** AIM arrives at the target proposition by applying only the definitions, lemmas, and rules encoded in the knowledge package. Throughout the process, AIM demonstrates a correct grasp of the derivation logic, faithfully applying the theory’s proof patterns to advance from premises to the theorem without relying on heuristic leaps.

During the proof of the homogenization problem, AIM proposed a property pertaining to the Cell Problem:  $\chi \in W^{1,\infty}(\omega)$ . Consequently, AIM successfully constructed a proof by applying Schauder theory, following some appropriate instruction.

**Prompt to AIM: Some prior knowledge of the entire Schauder theory**

**Content:** Lemma 1: Suppose  $\Omega_{\pm} = \mathbb{R}_{\pm}^d$ ,  $S = \{x_d = 0\}$ ,  $B_+ = \{x \in B(1) : x_d > 0\}$  and  $B_- = \{x \in B(1) : x_d < 0\}$ . Here  $B(1) = \{\|x\| \leq 1\}$ . Consider this equation: for  $V \in H_0^1(B(1); \mathbb{R}^d)$

$$(\nabla V : A_1 \nabla \tilde{\chi})_{B_+} + (\nabla V : A_2 \nabla \tilde{\chi})_{B_-} + (\tilde{r}, \nabla \cdot (aV))_{B_-} = 0,$$

$$\nabla \cdot (a\tilde{\chi}) = 0,$$

here  $\tilde{\chi} = D^\alpha \chi$ ,  $\tilde{r} = D^\alpha r$ ,  $|\alpha| \geq 1$  and  $A_1, A_2$  are constant tensors,  $a$  is a constant matrix. We have that  $\sum_{\pm} \|\chi\|_{H^k(B(\frac{1}{2}, \pm))} \leq C \|\chi\|_{L^2(B(1))}$  and  $\|r\|_{H^k(B(\frac{1}{2})_-)} \leq C \|r\|_{L^2(B(\frac{1}{2})_-)}$  for  $\forall k \geq 1$ .

**Response of AIM**

We follow the strategy already sketched in Lemma 5. Fix  $0 < \rho < t \leq 1/2$  and let  $(l^*, q^*) \in \mathcal{L}$  be an almost-minimizer at scale  $t$ , namely

$$E(t) \leq \left( \int_{B(t)} |\chi - l^*|^2 \right)^{1/2} + t \cdot |g_0 - \zeta(l^*, q^*)| + \left( \int_{B(t)_-} |r - q^*|^2 \right)^{1/2} + \varepsilon,$$

In this mode, AIM demonstrates coherent understanding and systematic application of both the provided lemmas and the ensuing proof strategy, thereby enhancing the probability of obtaining a complete and valid argument. And this process is amenable to formal verification.

## 4.3 INTERACTIVE ITERATION

Interactive Iteration is a paradigm for uncertain and complex problem settings built around a cyclic process: output  $\rightarrow$  human diagnosis and correction  $\rightarrow$  model optimization  $\rightarrow$  output  $\rightarrow \dots$ . By supplying the

necessary incremental information, performing condition checks, and validating with small examples, it avoids overcommitting to incorrect lines of reasoning and enables the reuse of validated fragments as stable modules. This iterative refinement helps decompose complex proofs into tractable components, thereby improving both efficiency and rigor. Through repeated cycles of adjustment, the proof is progressively elaborated and verified, ultimately yielding a complete and logically sound derivation.

- **Application Conditions:** We adopt an iterative interactive refinement mode when problem complexity precludes a one-shot solution, such as in multi-stage derivations, cross-domain dependencies, or tightly coupled constraints that necessitate phased exploration.
- **Typical Mathematical Scenarios:** In a typical scenario involving complex proofs, we often lack a clear and complete proof strategy and possess only preliminary derivation directions.
- **Expected Outcomes:** The expected outcomes of each iterative cycle are structured around error correction and direction guidance, forming a mechanism that drives the gradual refinement of proofs. AIM shall output a precisely targeted and revised derivation process. And based on AIM's revised results, we extract useful guidance content for the next iteration.

In the experiments related to error estimation, AIM was prompted with the error formulation. Analysis of its output revealed the following intermediate conclusions and derivations proposed by AIM:

#### Response of AIM

Then there exist  $\varepsilon_0 > 0$  and a constant  $C$ , depending only on  $d, \Omega, \|A\|_{H^\mu(Y)}$ , and the constructions of  $\eta_\varepsilon$  and  $S_\varepsilon$ , such that for every  $\psi \in H_0^1(\Omega; \mathbb{R}^d)$  and every  $\varepsilon \in (0, \varepsilon_0)$  one has the estimate

$$\left| \int_{\Omega} (\hat{A} - A(x/\varepsilon))(\nabla u_0 - \eta_\varepsilon S_\varepsilon^2(\nabla u_0)) : \nabla \psi \, dx \right| \leq C \left[ \varepsilon^{1/2} \|u_0\|_{H^2(\Omega)} + \|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \right] \|\nabla \psi\|_{L^2(\Omega)}.$$

#### Note

*The conclusion here is what we care about, and it also provides inspiration for our subsequent analysis and decomposition of the problem.*

While analyzing this derivation and the lemmas provided above, we summarized and formulated the following conclusion:

**Lemma 1.** *Suppose  $\Omega$  is a bounded Lipschitz domain in  $\mathbb{R}^d$ . Define the  $\varepsilon$ -neighborhood of the boundary as*

$$\Omega_t = \{x \in \Omega : \text{dis}(x, \partial\Omega) < t\}, \quad t > 0.$$

*For all  $\psi \in H_0^1(\Omega, \mathbb{R}^d)$ , the following estimate holds:*

$$\left| \int_{\Omega} A \nabla w_\varepsilon : \nabla \psi \, dx \right| \leq C \|\nabla \chi\|_{L^2(\Omega)} \left\{ \varepsilon \|S_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} + \|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \right\} + C \|\nabla \psi\|_{L^2(\Omega_{4\varepsilon})} \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})}.$$

We performed a certain amount of manual, straightforward derivation and determined that the following lemma is correct. Consequently, we provided this lemma to AIM, which then produced a correct derivation process.

#### 4.4 APPLICABILITY BOUNDARY AND EXCLUSION DOMAIN

For LLMs, some certain mathematical processes can be challenging to comprehend and execute accurately. Examples include handling intricate geometric configurations, deconstructing intermediate proof strategies, or addressing difficulties in mathematical derivations. Therefore, it is advisable to address these processes through human intervention.

441 For example, the cell problem serves as a bridge connecting the microscopic and macroscopic scales.  
 442 The primary difficulty for agents in such tasks lies in their insufficient ability to comprehend geometric  
 443 structures. In the experiment results of AIM, we observe a clear mathematical error:  
 444

#### 445 Response of AIM

$$446 \text{ in the elastic cell } Y_e : \quad \operatorname{div}_y (E y + \chi_e^E) = 0.$$

448 This error further leads to significant challenges for AIM in deriving a reasonable and correct form of the  
 449 homogenization equation. For such geometrically dependent constructive problems, direct human-driven  
 450 derivation and analysis may represent a more efficient and reliable approach.  
 451

#### 452 4.5 AUXILIARY OPTIMIZATION METHODS

453 Beyond core strategies such as **Direct Prompting** and **Interactive Iteration Refinement**, we employed  
 454 several supplementary techniques to enhance the reliability of AIM’s mathematical output.  
 455

- 457 • **Repeated Attempts and Proof Screening:** Owing to the inherent stochasticity of LLMs, the  
 458 same query often yields varying responses. By repeatedly testing identical problems, we filtered  
 459 the generated proofs and selected the most complete and coherent ones for further refinement.
- 460 • **Providing Target Conclusions to Improve Correctness:** We found that providing the target  
 461 conclusion, rather than posing fully open-ended proof tasks, materially improves correctness by  
 462 constraining the search space and guiding the reasoning path.
- 463 • **Model Selection Based on Task Requirements:** The choice of LLMs considerably affects per-  
 464 formance. Comparative analysis indicated that `o4-mini` excels in conceptual understanding and  
 465 constructing proof frameworks, while `DeepSeek-R1` is better suited for detailed mathematical  
 466 derivations and proof refinement.  
 467

468 Taken together, these auxiliary procedures substantially strengthen the AIM’s ability to produce mathemat-  
 469 ically valid and logically rigorous arguments and are integral to the overall methodology.  
 470

## 471 5 FAILURE MODES

472 AIM still exhibits several persistent and systematic failure modes when acting as a collaborative research  
 473 partner. These weaknesses are particularly evident in constructive and geometry-intensive tasks, where  
 474 success relies on rigorous interpretation of problem structures, verification of theorem preconditions, and  
 475 consistent integration between symbolic reasoning and geometric representation. In domains involving  
 476 constructive reasoning, AIM frequently struggles with assessing theorem preconditions and interpreting  
 477 geometric configurations, both of which remain unstable and error-prone. Moreover, the system lacks  
 478 robust mechanisms for self-checking preconditions and edge cases, leading to misjudged prerequisites,  
 479 breaks in reasoning chains, and internally inconsistent setups.  
 480

## 481 6 CONCLUSION

482 In this work, we employ AIM as a research partner to tackle a challenging problem in homogenization  
 483 theory. We investigate a human-AI collaboration paradigm that combines the computational strengths of AI  
 484 with the domain expertise and judgment of human researchers. Demonstrated through a homogenization  
 485 case study, this paradigm culminates in a rigorous proof of the target problem and reveals distinct interaction  
 486 patterns and empirical insights that may inform future AI-assisted mathematical research. These interaction  
 487 modes also illustrate how AI can extend the capability boundaries of human mathematicians.  
 488  
 489

## 490 REFERENCES

- 491 MAA invitational competitions, 2025a. URL <https://maa.org/maa-invitational-competitions/>.
- 492  
493
- 494 AIME problems and solutions, 2025b. URL [https://artofproblemsolving.com/wiki/index.php?title=AIME\\_Problems\\_and\\_Solutions](https://artofproblemsolving.com/wiki/index.php?title=AIME_Problems_and_Solutions).
- 495  
496
- 497 Introducing GPT-5, 2025. URL <https://openai.com/index/introducing-gpt-5/>.
- 498
- 499 Grok 4, 2025. URL <https://x.ai/news/grok-4>.
- 500
- 501 Introducing OpenAI o3 and o4-mini, 2025. URL <https://openai.com/index/introducing-o3-and-o4-mini/>.
- 502
- 503 Scott Aaronson and Freek Witteveen. Limits to black-box amplification in QMA, 2025. URL <https://arxiv.org/abs/2509.21131>.
- 504
- 505 Serena Maria Boccia. Schauder estimates for solutions of higher-order parabolic systems. *Methods and Applications of Analysis*, 20(1):47–68, 2013.
- 506
- 507 Yuri Chervonyi, Trieu H. Trinh, Miroslav Olšák, Xiaomeng Yang, Hoang Nguyen, Marcelo Menegali, Junehyuk Jung, Vikas Verma, Quoc V. Le, and Thang Luong. Gold-medalist performance in solving olympiad geometry with AlphaGeometry2, 2025. URL <https://arxiv.org/abs/2502.03544>.
- 508  
509  
510
- 511 Boris G. Galerkin. On a method of approximate solution of differential equations. *Zhurnal Russkogo Fiziko-Khimicheskogo Obshchestva imeni D.I. Mendeleeva*, 47(4):897–918, 1915.
- 512
- 513 Elliot Glazer, Ege Erdil, Tamay Besiroglu, Diego Chicharro, Evan Chen, Alex Gunning, Caroline Falkman Olsson, Jean-Stanislas Denain, Anson Ho, Emily de Oliveira Santos, Olli Järvinemi, Matthew Barnett, Robert Sandler, Matej Vrzala, Jaime Sevilla, Qiuyu Ren, Elizabeth Pratt, Lionel Levine, Grant Barkley, Natalie Stewart, Bogdan Grechuk, Tetiana Grechuk, Shreepranav Varma Enugandla, and Mark Wildon. FrontierMath: A benchmark for evaluating advanced mathematical reasoning in AI, 2025. URL <https://arxiv.org/abs/2411.04872>.
- 514  
515  
516  
517  
518
- 519 Anson Ho. Is AI already superhuman on FrontierMath?, 2025. URL <https://epoch.ai/gradient-updates/is-ai-already-superhuman-on-frontiermath>. Accessed: 2025-10-26.
- 520  
521
- 522 Olga A. Ladyzhenskaya, Vladimir A. Solonnikov, and Nina N. Ural'tseva. *Linear and Quasilinear Equations of Parabolic Type*. American Mathematical Society, Providence, RI, 1968.
- 523
- 524 Randall J. LeVeque. *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, USA, 2007.
- 525  
526  
527
- 528 Yuanhang Liu, Yanxing Huang, Yanqiao Wang, Peng Li, and Yang Liu. AI mathematician: Towards fully automated frontier mathematical research, 2025. URL <https://arxiv.org/abs/2505.22451>.
- 529
- 530 Thang Luong and Edward Lockhart. Advanced version of Gemini with Deep Think officially achieves gold-medal standard at the International Mathematical Olympiad, 2025. URL <https://deepmind.google/discover/blog/advanced-version-of-gemini-with-deep-think-officially-achieves-gold-medal-standard-at-the-international-mathematical-olympiad/>.
- 531  
532  
533  
534
- 535 Alexander Novikov, Ngãn Vũ, Marvin Eisenberger, Emilien Dupont, Po-Sen Huang, Adam Zsolt Wagner, Sergey Shirobokov, Borislav Kozlovskii, Francisco J. R. Ruiz, Abbas Mehrabian, M. Pawan Kumar, Abigail See, Swarat Chaudhuri, George Holland, Alex Davies, Sebastian Nowozin, Pushmeet Kohli, and Matej Balog. AlphaEvolve: A coding agent for scientific and algorithmic discovery, 2025. URL <https://arxiv.org/abs/2506.13131>.
- 536  
537  
538

539 QwenTeam. Qwen3-Max: Just scale it, 2025. URL [https://qwen.ai/blog?id=241398b9cd63](https://qwen.ai/blog?id=241398b9cd6353de490b0f82806c7848c5d2777d&from=research.latest-advancements-list)  
540 [53de490b0f82806c7848c5d2777d&from=research.latest-advancements-list](https://qwen.ai/blog?id=241398b9cd6353de490b0f82806c7848c5d2777d&from=research.latest-advancements-list).  
541  
542 Bernardino Romera-Paredes, Mohammadamin Barekatin, Alexander Novikov, Matej Balog, M. Pawan  
543 Kumar, Emilien Dupont, Francisco J. R. Ruiz, Jordan S. Ellenberg, Pengming Wang, Omar Fawzi,  
544 Pushmeet Kohli, and Alhussein Fawzi. Mathematical discoveries from program search with large  
545 language models. *Nature*, 625(7995):468–475, Jan 2024. ISSN 1476-4687. doi: 10.1038/s41586-023-  
546 06924-6. URL <https://doi.org/10.1038/s41586-023-06924-6>.  
547  
548 Simon Singh. *Fermat’s Last Theorem*. Fourth Estate, London, 1997.  
549  
550 Trieu H. Trinh, Yuhuai Wu, Quoc V. Le, He He, and Thang Luong. Solving olympiad geometry without  
551 human demonstrations. *Nature*, 625(7995):476–482, Jan 2024. ISSN 1476-4687. doi: 10.1038/s41586-  
552 023-06747-5. URL <https://doi.org/10.1038/s41586-023-06747-5>.  
553  
554 Jinping Zhuge. Regularity of a transmission problem and periodic homogenization. *Journal de Mathéma-*  
555 *tiques Pures et Appliquées*, 153:213–247, 2021. ISSN 0021-7824. doi: [https://doi.org/10.1016/j.matpur](https://doi.org/10.1016/j.matpur.2021.07.003)  
556 [.2021.07.003](https://doi.org/10.1016/j.matpur.2021.07.003). URL [https://www.sciencedirect.com/science/article/pii/S002](https://www.sciencedirect.com/science/article/pii/S0021782421001069)  
557 [1782421001069](https://www.sciencedirect.com/science/article/pii/S0021782421001069).  
558  
559  
560  
561  
562  
563  
564  
565  
566  
567  
568  
569  
570  
571  
572  
573  
574  
575  
576  
577  
578  
579  
580  
581  
582  
583  
584  
585  
586  
587

## A PROBLEM DESCRIPTION

The mathematical research problem we investigate in this work is an instance of a Stokes–Lamé transmission system with a vanishing fluid inclusion, analyzed in the homogenization regime  $\varepsilon \rightarrow 0$ . This problem will be referred to as the **Homogenization Problem** in the rest of this work.

Consider  $D \subset \Omega \subset \mathbb{R}^d$  ( $d \geq 2$ ), where  $\Omega$  is elastic material and  $D$  is the high contrast inclusion part.

- $\Omega$  is open bounded with connected  $C^\infty$  boundary  $\partial\Omega$ .
- $D$  is open, has a finite number of components and has a Lipschitz boundary  $\partial D$ .
- $\Omega \setminus D$  is connected with Lipschitz boundary  $\partial\Omega \cup \partial D$ . The connected components of  $D$  are enumerated as  $D_i$ ,  $i = 1, \dots, N$ ,  $N$  is finite.

And given  $\varepsilon \in (0, 1)$ ,  $D = D_\varepsilon$  is part of an  $\varepsilon$ -periodic array of small inclusions constructed as follows, in several steps.

$Y = (-\frac{1}{2}, \frac{1}{2})^d$  is the unit cell.  $\omega \subset Y$  is a simple connected open subset with connected Lipschitz boundary such that  $\text{dist}(\omega, \partial Y) > 0$ .  $Y_f = Y \setminus \bar{\omega}$  is the model environment in the unit scale.

Given  $\varepsilon > 0$  and  $\mathbf{n} \in \mathbb{Z}^d$ , we denote  $\varepsilon(\mathbf{n} + Y)$  and  $\varepsilon(\mathbf{n} + \omega)$  by  $Y_\varepsilon^{\mathbf{n}}$  and  $\omega_\varepsilon^{\mathbf{n}}$ , respectively. Let  $\Pi_\varepsilon$  be the set of lattice points  $\mathbf{n}$  such that  $\bar{Y}_\varepsilon^{\mathbf{n}}$  be contained in  $\Omega$ , i.e.,

$$\Pi_\varepsilon := \left\{ \mathbf{n} \in \mathbb{Z}^d : \bar{Y}_\varepsilon^{\mathbf{n}} \subset \Omega \right\}, \quad (5)$$

then the inclusions set  $D = D_\varepsilon$  and the background part  $\Omega_\varepsilon$  are defined by

$$D_\varepsilon := \bigcup_{\mathbf{n} \in \Pi_\varepsilon} \omega_\varepsilon^{\mathbf{n}} \quad \Omega_\varepsilon := \Omega \setminus \bar{D}_\varepsilon. \quad (6)$$

A pair of real numbers  $(\lambda, \mu)$  is called admissible and referred to as a Lamé pair, if they satisfy  $\mu > 0$  and  $d\lambda + 2\mu > 0$ . For a Lamé pair  $(\lambda, \mu)$ , the elastostatic system (Lamé system) reads

$$\mathcal{L}_{\lambda, \mu} u := \mu \Delta u + (\lambda + \mu) \nabla \text{div} u, \quad (7)$$

where  $u = (u^1, \dots, u^d)$  represents the displacement field and the divergence of  $u$  is given by  $\text{div} u = \sum_{i=1}^d \frac{\partial u^i}{\partial x_i}$ . **It is worth noting that  $u$ , rather than  $\mathbf{u}$ , is used here to ensure consistency with the notation adopted by AIM.** The Lamé operator can be written as  $\nabla \cdot \sigma(u)$  where

$$\sigma(u) := \lambda(\nabla \cdot u) \mathbb{I}_d + 2\mu \mathcal{D}(u), \quad (8)$$

$$\mathcal{D}(u) = \frac{1}{2}(\nabla + \nabla^T)u = \frac{1}{2}(\partial_i u^j + \partial_j u^i)_{ij}. \quad (9)$$

The corresponding conormal derivative (boundary traction) at the boundary of a domain  $E$  is

$$\left. \frac{\partial u}{\partial \nu_{(\lambda, \mu)}} \right|_{\partial E} := \sigma(u)N = \lambda(\text{div} u)N + 2\mu \mathcal{D}(u)N \quad \text{on } \partial E. \quad (10)$$

We consider the space  $\mathbb{R}$  of rigid motions in  $\mathbb{R}^d$ , defined by

$$\mathbb{R} := \left\{ \mathbf{r} = (r_1, \dots, r_d)^T : \mathcal{D}(\mathbf{r}) = 0 \text{ in } \mathbb{R}^d \right\}.$$

We define  $H_{\mathbb{R}}^{-\frac{1}{2}}(\partial D_\varepsilon)$  as the subspace of  $H^{-\frac{1}{2}}(\partial D_\varepsilon)$  that is orthogonal to  $\mathbb{R}$ , i.e.,

$$H_{\mathbb{R}}^{-\frac{1}{2}}(\partial D_\varepsilon) := \left\{ \phi \in H^{-\frac{1}{2}}(\partial D_\varepsilon) : (\phi, \mathbf{r})_{(H^{\frac{1}{2}}(\partial D_{\varepsilon_i}), H^{-\frac{1}{2}}(\partial D_{\varepsilon_i}))} = 0, \forall \mathbf{r} \in \mathbb{R} \text{ and } 1 \leq i \leq N \right\}. \quad (11)$$

637 Consider the displacement field  $u_\varepsilon$  satisfying the following transmission system:

$$\begin{cases}
\mathcal{L}_{\lambda,\mu}u_\varepsilon = 0 & \text{in } \Omega \setminus \overline{D_\varepsilon}, \\
\mathcal{L}_{\tilde{\lambda},\tilde{\mu}}u_\varepsilon = 0 & \text{in } D_\varepsilon, \\
u_\varepsilon|_- = u_\varepsilon|_+ \text{ and } \frac{\partial u_\varepsilon}{\partial \nu(\tilde{\lambda},\tilde{\mu})}\Big|_- = \frac{\partial u_\varepsilon}{\partial \nu(\lambda,\mu)}\Big|_+ & \text{on } \partial D_\varepsilon, \\
\frac{\partial u_\varepsilon}{\partial \nu(\lambda,\mu)}\Big|_{\partial\Omega} = g \in H_{\mathbb{R}}^{-\frac{1}{2}}(\partial\Omega) \text{ and } u_\varepsilon|_{\partial\Omega} \in H_{\mathbb{R}}^{\frac{1}{2}}(\partial\Omega). & 
\end{cases} \quad (12)$$

645 Suppose  $\tilde{\mu}$  fixed, then we arrive at the equations about the **incompressible inclusion limit**. In this case,  
646 the transmission problem is a coupled **Lamé-Stokes system**:

$$\begin{cases}
\mathcal{L}_{\lambda,\mu}u_\varepsilon = 0 & \text{in } \Omega \setminus \overline{D_\varepsilon}, \\
\mathcal{L}_{\tilde{\mu}}(u_\varepsilon, p_\varepsilon) = 0 \text{ and } \operatorname{div} u_\varepsilon = 0 & \text{in } D_\varepsilon, \\
u_\varepsilon|_- = u_\varepsilon|_+ \text{ and } \frac{\partial(u_\varepsilon, p_\varepsilon)}{\partial \nu(\infty, \tilde{\mu})}\Big|_- = \frac{\partial u_\varepsilon}{\partial \nu(\lambda, \mu)}\Big|_+ & \text{on } \partial D_\varepsilon, \\
\frac{\partial u_\varepsilon}{\partial \nu(\lambda, \mu)}\Big|_{\partial\Omega} = g \in H_{\mathbb{R}}^{-\frac{1}{2}}(\partial\Omega) \text{ and } u_\varepsilon|_{\partial\Omega} \in H_{\mathbb{R}}^{\frac{1}{2}}(\partial\Omega). & 
\end{cases} \quad (13)$$

654 Here,  $\mathcal{L}_{\tilde{\mu}}(u_\varepsilon, p_\varepsilon) = \tilde{\mu}\Delta u_\varepsilon + \nabla p_\varepsilon$  denotes the Stokes operator with viscosity constant  $\tilde{\mu}$ , and  $p_\varepsilon$  is the  
655 pressure field.  $N$  is the outward unit normal vector to the boundary of the domain. Its exterior derivative is  
656 defined as  $\frac{\partial(u,p)}{\partial \nu(\infty, \mu)}\Big|_- := pN + 2\mu D(u)N$ .

658 We need to conclude the limit homogenization equation as the scale of the cell tends to be zero  $\varepsilon \rightarrow 0$ . At  
659 the same time, we wonder the estimate between the original solution  $u_\varepsilon$  and the limited solution  $u_{\text{lim}}$ , i.e.,

$$660 \quad \|u_\varepsilon - u_{\text{lim}}\|_{H^1(\Omega)} \lesssim \varepsilon^\alpha \quad (14)$$

662 for some  $\alpha \in (0, 1)$ . It is necessary to analyze and obtain determined value of  $\alpha$ , and strictly prove this  
663 conclusion.

## B DETAILED CASE STUDIES

### B.1 DIRECT PROMPTING — THEOREM PROMPTS

Here are part of the experimental results on error estimation output by AIM after we directly prompted it with the content of some theorems. The response is a complete lemma and its proof process generated by AIM, from which we can observe how AIM utilized the prompted content and the details of its deduction.

#### Prompt to AIM

##### Disclaimer

For clarity, we format prompts using typographic styles (e.g., bold text) and render formulas as symbols rather than raw LaTeX code, while the content remains consistent with that of the actual experiment. “**Content**” is one of the labels used by AIM to distinguish different components and can be safely ignored when interpreting the prompt. The same applies below.

##### Content:

You can use the following theorem to analyze the problem, integrating the content of the theorem and its derivation process.

##### Disclaimer

The outputs of AIM include content such as `\begin{lemma}\end{lemma}`. We do not modify these outputs. LaTeX automatically assigns the numbering. Please ignore the numbers and the same applies below.

**Lemma 2.** Let  $A_1, A_2$  be constant tensors and  $a$  be a constant matrix. Consider the system:

$$\begin{cases} \nabla \cdot (A_1 \nabla \chi) = 0 & \text{in } B(1)_+, \\ \nabla \cdot (A_2 \nabla \chi) + \nabla \cdot (a^T r) = 0 & \text{in } B(1)_-, \\ \nabla \cdot (a \chi) = 0 & \text{in } B(1)_-, \\ \chi|_+ = \chi|_- & \text{on } B(1) \cap \{x_d = 0\}, \\ \left. \frac{\partial \chi}{\partial \nu} \right|_+ - \left. \frac{\partial \chi}{\partial \nu} \right|_- = g_0 & \text{on } B(1) \cap \{x_d = 0\}. \end{cases}$$

where  $\chi, r$  are weak solutions.

Moreover, for all  $\rho \in (0, t)$ , integrating the above inequalities yields:

$$\left( \int_{B(\rho)} |\chi - l|^2 \right)^{\frac{1}{2}} + \rho |g_0 - \zeta(l, q)| \leq \left| \frac{\rho}{t} \right|^{\beta+1} \left( \left( \int_{B(t)} |\chi|^2 \right)^{\frac{1}{2}} + t |g_0| \right)$$

##### Note

This derivation process yields a crucial conclusion and embodies the inherent logic of the inferential steps

Thus, for all  $(l', q') \in \mathcal{L}$ , by the inequality above, we have:

$$\begin{aligned} & \inf_{(l, q) \in \mathcal{L}} \left\{ \left( \int_{B(\rho)} |\chi - l|^2 \right)^{\frac{1}{2}} + \rho |g_0 - \zeta(l, q)| \right\} \\ & \leq C \left| \frac{\rho}{t} \right|^{\beta+1} \inf_{(l, q) \in \mathcal{L}} \left( \left( \int_{B(t)} |\chi - l|^2 \right)^{\frac{1}{2}} + t |g_0 - \zeta(l, q)| \right) \end{aligned}$$

Further, for all  $(l', q') \in \mathcal{L}$ , it follows that:

$$\begin{aligned} & \inf_{(l, q) \in \mathcal{L}} \left\{ \left( \int_{B(\rho)} |\chi - l|^2 \right)^{\frac{1}{2}} + \rho |g_0 - \zeta(l, q)| \right\} \\ & \leq C \left| \frac{\rho}{t} \right|^{\beta+1} \inf_{(l, q) \in \mathcal{L}} \left( \left( \int_{B(t)} |\chi - l|^2 \right)^{\frac{1}{2}} + t |g_0 - \zeta(l, q)| \right) \end{aligned}$$

## Response of AIM

### Disclaimer

For clarity, we render formulas as symbols rather than raw LaTeX code. The same applies below.

**Lemma 3.** Let  $u = w_t - l$  and  $v = s_t - q$ , where  $l$  and  $q$  are the piecewise linear solutions from Lemma 4. Then, under the given hypotheses, there exists a constant  $C > 0$  independent of  $t$  such that:

$$\int_{B(t)} |\nabla w_t - \nabla \chi|^2 dx + \int_{B(t)_-} |r - s_t|^2 dx \leq C \left( t^{2\beta} \int_{B(t)} |\nabla \chi|^2 dx + t^{2\beta} \int_{B(t)_-} |r|^2 dx \right),$$

where  $\beta \in (0, 1)$  is the Hölder exponent from Lemma 4.

**correctness:** True

### Note

“**correctness:** True” is generated and used internally by AIM, and can be safely ignored when interpreting the prompt. The same applies to similar cases below.

*Proof.* **\*\*Step 1: Apply Lemma 8\*\*** By Lemma 8, there exists  $C_1 > 0$  such that:

$$\int_{B(t)} |\nabla u|^2 dx + \int_{B(t)_-} |v|^2 dx \leq C_1 \left( t^{-2} \int_{B(t)} |\chi - l|^2 dx + \int_{B(t)_-} |r - q|^2 dx \right).$$

**\*\*Step 2: Bounding  $\int_{B(t)} |\chi - l|^2 dx$ \*\*** From Lemma 4 and scaling arguments (see hint-5), we derive:

$$\int_{B(t)} |\chi - l|^2 dx \leq C_2 t^{2(\beta+1)} \int_{B(t)} |\nabla \chi|^2 dx.$$

**\*\*Step 3: Bounding  $\int_{B(t)_-} |r - q|^2 dx$ \*\*** Similarly, Lemma 4 gives:

$$\int_{B(t)_-} |r - q|^2 dx \leq C_3 t^{2\beta} \int_{B(t)_-} |r|^2 dx.$$

**\*\*Step 4: Expanding Key Terms\*\*** Decompose the left-hand side of the conjecture:

$$\int_{B(t)} |\nabla w_t - \nabla \chi|^2 dx = \int_{B(t)} |\nabla u + (\nabla l - \nabla \chi)|^2 dx,$$

$$\int_{B(t)_-} |r - s_t|^2 dx = \int_{B(t)_-} |(r - q) - v|^2 dx.$$

Expanding these squares:

$$\begin{aligned} |\nabla u + (\nabla l - \nabla \chi)|^2 & \leq 2|\nabla u|^2 + 2|\nabla l - \nabla \chi|^2, \\ |(r - q) - v|^2 & \leq 2|r - q|^2 + 2|v|^2. \end{aligned}$$

784  
785  
786  
787  
788  
789  
790  
791  
792  
793  
794  
795  
796  
797  
798  
799  
800  
801  
802  
803  
804  
805  
806  
807  
808  
809  
810  
811  
812  
813  
814  
815  
816  
817  
818  
819  
820  
821  
822  
823  
824  
825  
826  
827  
828  
829  
830  
831  
832

Integrate and sum these inequalities:

$$\begin{aligned} & \int_{B(t)} |\nabla w_t - \nabla \chi|^2 dx + \int_{B(t)_-} |r - s_t|^2 dx \\ & \leq 2 \left( \int_{B(t)} |\nabla u|^2 dx + \int_{B(t)_-} |v|^2 dx \right) + 2 \left( \int_{B(t)} |\nabla l - \nabla \chi|^2 dx + \int_{B(t)_-} |r - q|^2 dx \right). \end{aligned}$$

**\*\*Step 5: Bounding Residual Terms\*\*** From Lemma 4 and scaling:

$$\int_{B(t)} |\nabla l - \nabla \chi|^2 dx \leq C_4 t^{2\beta} \int_{B(t)} |\nabla \chi|^2 dx.$$

**\*\*Step 6: Combining All Estimates\*\*** Substitute Lemma 8 and Steps 2-5 into Step 4:

$$\text{LHS} \leq 2C_1 \left( t^{-2} C_2 t^{2(\beta+1)} \|\nabla \chi\|^2 + C_3 t^{2\beta} \|r\|^2 \right) + 2 \left( C_4 t^{2\beta} \|\nabla \chi\|^2 + C_3 t^{2\beta} \|r\|^2 \right).$$

Simplify exponents and constants:

$$\text{LHS} \leq C \left( t^{2\beta} \|\nabla \chi\|^2 + t^{2\beta} \|r\|^2 \right),$$

where  $C = 2C_1(C_2 + C_3) + 2(C_4 + C_3)$ . This completes the proof.  $\square$

## B.2 DIRECT PROMPTING — CONCEPTUAL GUIDANCE

Here is the prompt and output content of AIM in the error estimation section. After we guided the specific form of error estimation, this is the content of a key lemma and its proof process generated by AIM.

Based on this proof process and combined with the interactive iteration mode, we subsequently developed a clearer proof framework and guidance content.

### Prompt to AIM

**Content:** Consider the following form of the error.

Define the cut-off function as follows:  $\eta_\varepsilon \in C_0^\infty(\Omega)$ , and

$$\begin{aligned} 0 & \leq \eta_\varepsilon \leq 1, \quad |\nabla \eta_\varepsilon| \leq \frac{C}{\varepsilon}, \\ \eta_\varepsilon & = 1, \quad \text{when } \text{dist}(x, \partial\Omega) \geq 4\varepsilon, \\ \eta_\varepsilon & = 0, \quad \text{when } \text{dist}(x, \partial\Omega) \leq 3\varepsilon. \end{aligned}$$

And we also set this convolution operator:

$$S_\varepsilon(f)(x) = \rho_\varepsilon * f(x) = \int_{\mathbb{R}^d} f(x-y) \rho_\varepsilon dy,$$

Here  $\rho \in C_0^\infty(B(0, \frac{1}{2}))$ ,  $\rho \geq 0$ , and  $\int_{\mathbb{R}^d} \rho dx = 1$ .

Suppose

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi \left( \frac{x}{\varepsilon} \right) \eta_\varepsilon S_\varepsilon^2(\nabla u_0).$$

### Response of AIM

#### Lemma 4.

**Conjecture 1.** Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain,  $A(y)$  a 1-periodic, uniformly elliptic, bounded coefficient in  $H^\mu(Y)$  with  $\mu > d/2$ , and  $\chi(y)$  the associated  $H^1$ -cell-corrector. Let

833  $u_0 \in H^2(\Omega)$  and, for  $\varepsilon > 0$ , let  $S_\varepsilon$  be the Friedrichs-mollifier and  $\eta_\varepsilon \in C_0^\infty(\Omega)$  a cut-off with

$$834 \quad \eta_\varepsilon \equiv 1 \text{ on } \Omega \setminus \Omega_{4\varepsilon}, \quad \eta_\varepsilon \equiv 0 \text{ on } \Omega_{5\varepsilon}, \quad |\nabla \eta_\varepsilon| \leq C/\varepsilon.$$

835 Define the ‘‘corrector-mismatch’’ term

$$836 \quad J(\psi) = \int_{\Omega} M(x/\varepsilon) (\eta_\varepsilon S_\varepsilon^2(\nabla u_0) - \nabla u_0) : \nabla \psi \, dx,$$

837 where  $M(y) = \nabla \chi(y)A(y) + \nabla \chi(y)A(y)\nabla \chi(y)$  is bounded and periodic. Then there is  $C$  independent of  $\varepsilon$ ,  $u_0$ ,  $\psi$  such that for every  $\psi \in H_0^1(\Omega; \mathbb{R}^d)$  and all  $0 < \varepsilon < \varepsilon_0$ ,

$$838 \quad |J(\psi)| \leq C [\|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} + \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})}] \|\nabla \psi\|_{L^2(\Omega)}.$$

839 **correctness:** True

840 *Proof.* *Proof.* We set

$$841 \quad M(y) = \nabla_y \chi(y) A(y) + \nabla_y \chi(y) A(y) \nabla_y \chi(y),$$

842 so that by periodicity and the cell-problem  $M \in L^\infty(Y)$ . Denote

$$843 \quad J(\psi) = \int_{\Omega} M(x/\varepsilon) (\eta_\varepsilon S_\varepsilon^2(\nabla u_0) - \nabla u_0) : \nabla \psi \, dx.$$

844 We split  $\Omega$  into the ‘‘interior’’ region  $D = \text{int}(\Omega \setminus \Omega_{5\varepsilon})$  and the ‘‘boundary collar’’  $C = \Omega_{5\varepsilon}$ . Since  $\eta_\varepsilon \equiv 1$  on  $D$  and  $0 \leq \eta_\varepsilon \leq 1$  on  $C$ , we estimate each piece separately.

845 1. Interior region  $D = \Omega \setminus \Omega_{5\varepsilon}$ . Here  $\eta_\varepsilon \equiv 1$ , so

$$846 \quad \eta_\varepsilon S_\varepsilon^2(\nabla u_0) - \nabla u_0 = S_\varepsilon^2(\nabla u_0) - \nabla u_0.$$

847 Using  $|M| \leq C$  and Cauchy–Schwarz,

$$848 \quad |J_D| \leq C \int_D |S_\varepsilon^2(\nabla u_0) - \nabla u_0| |\nabla \psi| \leq C \|S_\varepsilon^2(\nabla u_0) - \nabla u_0\|_{L^2(D)} \|\nabla \psi\|_{L^2(D)}.$$

849 Since  $D \subset \Omega \setminus \Omega_{2\varepsilon}$ , the standard mollifier-estimate (Lemma 5) yields

$$850 \quad \|S_\varepsilon^2(\nabla u_0) - \nabla u_0\|_{L^2(D)} \leq \|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})},$$

851 and clearly  $\|\nabla \psi\|_{L^2(D)} \leq \|\nabla \psi\|_{L^2(\Omega)}$ . Hence

$$852 \quad |J_D| \leq C \|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \|\nabla \psi\|_{L^2(\Omega)}.$$

853 2. Boundary-collar  $C = \Omega_{5\varepsilon}$ . On  $C$  we only know  $0 \leq \eta_\varepsilon \leq 1$ , so

$$854 \quad |\eta_\varepsilon S_\varepsilon^2(\nabla u_0) - \nabla u_0| \leq |S_\varepsilon^2(\nabla u_0)| + |\nabla u_0| \leq C |\nabla u_0|$$

855 (by boundedness of the mollifier in  $L^2$ ). Hence

$$856 \quad |J_C| \leq C \int_C |\nabla u_0| |\nabla \psi| \leq C \|\nabla u_0\|_{L^2(C)} \|\nabla \psi\|_{L^2(C)} = C \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} \|\nabla \psi\|_{L^2(\Omega_{5\varepsilon})}.$$

857 Finally, since  $\Omega_{5\varepsilon} \subset \Omega$  we have  $\|\nabla \psi\|_{L^2(\Omega_{5\varepsilon})} \leq \|\nabla \psi\|_{L^2(\Omega)}$ , so

$$858 \quad |J_C| \leq C \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} \|\nabla \psi\|_{L^2(\Omega)}.$$

859 Combining  $|J| \leq |J_D| + |J_C|$  gives

$$860 \quad |J(\psi)| \leq C [\|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} + \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})}] \|\nabla \psi\|_{L^2(\Omega)},$$

861 which is the claimed estimate. □

## B.3 COORDINATED APPLICATION OF SCHAUDER THEORY

This example illustrates the use of the theory-coordinated application paradigm to solve cell problems. We first provided theoretical lemmas and definitions as foundational guidance, which in turn instructed AIM to generate the corresponding final theorem conclusions.

## Prompt to AIM

**Content:**

**Lemma. 1:** Suppose  $\Omega_{\pm} = \mathbb{R}_{\pm}^d$ ,  $S = \{x_d = 0\}$ ,  $B_+ = \{x \in B(1) : x_d > 0\}$  and  $B_- = \{x \in B(1) : x_d < 0\}$ . Here  $B(1) = \{\|x\| \leq 1\}$ . Consider this equation: for  $V \in H_0^1(B(1); \mathbb{R}^d)$

$$\begin{aligned} (\nabla V : A_1 \nabla \tilde{\chi})_{B_+} + (\nabla V : A_2 \nabla \tilde{\chi})_{B_-} + (\tilde{r}, \nabla \cdot (aV))_{B_-} &= 0, \\ \nabla \cdot (a\tilde{\chi}) &= 0, \end{aligned}$$

where  $\tilde{\chi} = D^\alpha \chi$ ,  $\tilde{r} = D^\alpha r$  ( $|\alpha| \geq 1$ ),  $A_1, A_2$  are constant tensors, and  $a$  is a constant matrix. We have that for  $\forall k \geq 1$

$$\sum_{\pm} \|\chi\|_{H^k(B(\frac{1}{2})_{\pm})} \leq C \|\chi\|_{L^2(B(1))} \quad \text{and} \quad \|r\|_{H^k(B(\frac{1}{2})_-)} \leq C \|r\|_{L^2(B(\frac{1}{2})_-)}$$

**Lemma. 2:** Suppose that  $M_{\pm}$  is the constant matrix in  $\mathbb{R}^{d \times d}$ , the following are equivalent:

$$\begin{aligned} \forall y \in \{y_d = 0\} \quad M_+ x &= M_- x, \\ \exists c \in \mathbb{R}^d, \text{ s.t. } M_+ - M_- &= c e_d^T, \\ (I - e_d^T e_d) M_+ &= (I - e_d^T e_d) M_-. \end{aligned}$$

**Definition.**  $A_1, A_2$  are constant tensors,  $a$  is a constant matrix. If  $M_{\pm}$  satisfy the above lemma, and  $\nabla \cdot (aM_- y) = 0$  in  $B(t)_-$ .

We let  $l(y) = M_+ y \mathbf{1}_{y_d \geq 0} + M_- y \mathbf{1}_{y_d \leq 0} + C$ ,  $q(y) = r(0)$ . We call  $l, q$  are the piecewise linear solutions of the following equation:

$$\begin{aligned} \nabla \cdot (A_1 \nabla l) &= 0 && \text{in } B(t)_+, \\ \nabla \cdot (A_2 \nabla l) + \nabla \cdot (a^T q) &= 0, \quad \nabla \cdot (al) = 0 && \text{in } B(t)_-, \\ l_+ = l_-, \quad \frac{\partial l}{\partial \nu} \Big|_+ - \frac{\partial l}{\partial \nu} \Big|_- &= (A_1 M_+) e_d - (A_2 M_- + a^T r(0)) e_d && \text{on } \partial B(t). \end{aligned}$$

Suppose that  $\mathcal{L}$  is the space of all the piecewise linear solutions of the above equation. And  $\forall (l, q) \in \mathcal{L}$ , we define  $\zeta(l, q) = \frac{\partial l}{\partial \nu} \Big|_+ - \frac{\partial l}{\partial \nu} \Big|_-$ .

**Lemma. 3:**  $A_1, A_2$  are constant tensors,  $a$  is a constant matrix.

$$\begin{aligned} \nabla \cdot (A_1 \nabla \chi) &= 0 && \text{in } B(1)_+, \\ \nabla \cdot (A_2 \nabla \chi) + \nabla \cdot (a^T r) &= 0, \quad \nabla \cdot (a\chi) = 0 && \text{in } B(1)_-, \\ \chi_+ = \chi_-, \quad \frac{\partial \chi}{\partial \nu} \Big|_+ - \frac{\partial \chi}{\partial \nu} \Big|_- &= g_0 && \text{on } B(1) \cap \{x_d = 0\}. \end{aligned}$$

$\chi, r$  are the weak solutions of the above equation. Then for  $\forall k \geq 0$ ,  $\alpha \in [0, 1]$ , we have

$$\sum_{\pm} \|\chi\|_{H^k(B(\frac{1}{2})_{\pm})} \leq C \left( \|\chi\|_{L^2(B(1))} + |g_0| \right)$$

**Lemma. 4:**  $A_1, A_2$  are constant tensors,  $a$  is a constant matrix.

$$\begin{aligned} \nabla \cdot (A_1 \nabla \chi) &= 0 && \text{in } B(1)_+, \\ \nabla \cdot (A_2 \nabla \chi) + a^T \nabla r &= 0, \quad \nabla \cdot (a\chi) = 0 && \text{in } B(1)_-, \\ \chi_+ = \chi_-, \quad \frac{\partial \chi}{\partial \nu} \Big|_+ - \frac{\partial \chi}{\partial \nu} \Big|_- &= g_0 && \text{on } B(1) \cap \{x_d = 0\}. \end{aligned}$$

$\chi, r$  are the weak solutions of the above equation. And we let  $l(y) = (\nabla \chi)_+(0)y \mathbf{1}_{y_d \geq 0} + (\nabla \chi)_-(0)y \mathbf{1}_{y_d \leq 0} + \chi(0)$ ,  $q(y) = r(0)$ .

By Lemma 1 we know that  $\chi(0), (\nabla \chi)_\pm(0)$  are well-defined, and  $(I - e_d^T e_d)(\nabla \chi)_+ = (I - e_d^T e_d)(\nabla \chi)_-$  on  $B(t) \cap \{y_d = 0\}$ . So by Lemma 2, we know  $(l, q) \in \mathcal{L}$ .

Thus for some  $\beta \in (0, 1)$  and  $\forall y \in B(\frac{1}{2})$ :

$$\begin{aligned} |\chi(y) - l(y)| &\leq |\chi(y) - \chi(0) - (\nabla \chi)_\pm(0)y| \\ &\leq C |y|^{\beta+1} [\chi]_{C^{1,\beta}(B(\frac{1}{2})_\pm)} \\ &\leq C |y|^{\beta+1} \left( \left( \int_{B(1)} |\chi|^2 \right)^{\frac{1}{2}} + |g_0| \right) \end{aligned}$$

and for  $\forall y \in B(\frac{1}{2})_-$ :

$$|r - q| \leq C |y|^\beta [r]_{C^{0,\beta}(B(\frac{1}{2})_-)} \leq C |y|^\beta \left( \int_{B(1)_-} |r|^2 \right)^{\frac{1}{2}}.$$

Therefore, for some  $\beta \in (0, 1)$  and  $\forall y \in B(\frac{t}{2})$ :

$$\begin{aligned} |\chi(y) - l(y)| &\leq |\chi(y) - \chi(0) - (\nabla \chi)_\pm(0)y| \\ &\leq C \left| \frac{y}{t} \right|^{\beta+1} [\chi]_{C^{1,\beta}(B(\frac{t}{2})_\pm)} \\ &\leq C \left| \frac{y}{t} \right|^{\beta+1} \left( \left( \int_{B(t)} |\chi|^2 \right)^{\frac{1}{2}} + t |g_0| \right) \end{aligned}$$

and for  $\forall y \in B(\frac{t}{2})_-$ :

$$|r - q| \leq C \left| \frac{y}{t} \right|^\beta [r]_{C^{0,\beta}(B(\frac{t}{2})_-)} \leq C \left| \frac{y}{t} \right|^\beta \left( \int_{B(t)_-} |r|^2 \right)^{\frac{1}{2}}.$$

**Lemma. 5:**  $A_1, A_2$  are constant tensors,  $a$  is a constant matrix.

$$\begin{aligned} \nabla \cdot (A_1 \nabla \chi) &= 0 && \text{in } B(1)_+; \\ \nabla \cdot (A_2 \nabla \chi) + \nabla \cdot (a^T r) &= 0, \quad \nabla \cdot (a\chi) = 0 && \text{in } B(1)_-; \\ \chi_+ = \chi_-, \quad \frac{\partial \chi}{\partial \nu} \Big|_+ - \frac{\partial \chi}{\partial \nu} \Big|_- &= g_0 && \text{on } B(1) \cap \{x_d = 0\}. \end{aligned}$$

$\chi, r$  are the weak solutions of the above equation. Moreover,  $\forall \rho \in (0, t)$  integrate the above inequalities to get

$$\left( \int_{B(\rho)} |\chi - l|^2 \right)^{\frac{1}{2}} + \rho |g_0 - \zeta(l, q)| \leq \left| \frac{\rho}{t} \right|^{\beta+1} \left( \left( \int_{B(t)} |\chi|^2 \right)^{\frac{1}{2}} + t |g_0 - \zeta(l, q)| \right)$$

980 So  $\forall (l', q') \in \mathcal{L}$ , by the inequality above, we have

$$981 \text{inf}_{l, q \in \mathcal{L}} \left\{ \left( \int_{B(\rho)} |\chi - l|^2 \right)^{\frac{1}{2}} + \rho |g_0 - \zeta(l, q)| \right\} \leq C \left| \frac{\rho}{t} \right|^{\beta+1} \text{inf}_{l, q \in \mathcal{L}} \left( \left( \int_{B(t)} |\chi - l|^2 \right)^{\frac{1}{2}} + t |g_0 - \zeta(l, q)| \right)$$

982  
983  
984  
985  
986 **Lemma. 6:** Suppose  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a non-decreasing non-negative function satisfying  $\phi(\rho) \leq$   
987  $C \left( \frac{\rho}{t} \right)^\beta \phi(r) + Br^\alpha$ , where  $\beta > \alpha > 0$ ,  $C > 0$ .  
988 Then  $\forall 0 < \rho < r < R$ ,  $\exists C_1$ , s.t.  $\phi(\rho) \leq C_1 \left( \frac{\rho}{t} \right)^\alpha \phi(r) + B\rho^\alpha$ .  
989

## 990 Response of AIM

### 991 Lemma 5.

992 **Conjecture 2.** (Two-scale Campanato-decay for the excess) Let  $\chi \in H^1(B_1; \mathbb{R}^m)$  be the weak  
993 solution of the flat-interface two-phase conductivity–transmission system with constant coefficients  
994  $A_1, A_2, a$  and jump  $g_0$  as in Lemma 12, and define for each  $0 < t \leq \frac{1}{2}$  the “excess”  
995

$$996 \Phi(t) := \inf_{l \in \mathcal{L}} \left\{ \int_{B(t)} |\chi - l|^2 dx + t^{d+2} |\zeta(l) - g_0|^2 \right\},$$

997 where  $\mathcal{L}$  is the finite-dimensional family of piecewise-affine maps on  $B_1$  with tangential-continuity,  
998 and  $\zeta(l)$  its normal-flux-jump. Finally set  $\Psi \left( \frac{1}{2} \right) := \int_{B(\frac{1}{2})} |\nabla \chi|^2 dx + \int_{B(\frac{1}{2})_-} |\nabla r|^2 dx$  as in Lemma  
999 12.

1000 Then for every  $\beta \in (\alpha, 1)$  (where  $\alpha > 0$  is the exponent from the one-sided  $C^{1, \alpha/2}$ -regularity) and  
1001 every pair  $0 < \rho < t \leq \frac{1}{2}$  there exists  $C = C(d, \lambda, \Lambda, \beta) < \infty$  such that  
1002

$$1003 \Phi(\rho) \leq C \cdot \left( \frac{\rho}{t} \right)^{d+2+2\beta} \cdot \Phi(t) + C \cdot t^{d+2+\alpha} \cdot \Psi \left( \frac{1}{2} \right).$$

1004 **correctness:** True

1005 *Proof.* We split the proof into two main steps: (1) a sharp two-scale estimate for an almost-minimizer  
1006 at scale  $t$ , and (2) a trivial competitor at the smaller scale to capture the lower-order remainder term.  
1007 Step 1. (Decay from scale  $t$  to  $\rho$  by Campanato-type estimate) By definition of  $\Phi(t)$  there exists  
1008  $l_t \in \mathcal{L}$  such that

$$1009 \int_{B(t)} |\chi - l_t|^2 + t^{d+2} |\zeta(l_t) - g_0|^2 \leq 2\Phi(t).$$

1010 Set  $u := \chi - l_t$ . Then  $u$  solves the same two-phase system, but with normal-flux-jump  $g_0 - \zeta(l_t)$ .  
1011 Hence we may invoke Lemma 5 (the Campanato-type decay with flux-control). For every  $0 < \rho < t$   
1012 it yields

$$1013 \left( \int_{B(\rho)} |u|^2 \right)^{1/2} + \rho \cdot |\zeta(l_t) - g_0| \leq C \cdot (\rho/t)^{1+\beta} \cdot \left[ \left( \int_{B(t)} |u|^2 \right)^{1/2} + t \cdot |\zeta(l_t) - g_0| \right].$$

1014 Squaring, multiplying by the volume factors  $\rho^d$  and  $t^d$  and absorbing constants gives

$$1015 \int_{B(\rho)} |u|^2 + \rho^{d+2} |\zeta(l_t) - g_0|^2 \leq C \cdot (\rho/t)^{d+2+2\beta} \cdot \left[ \int_{B(t)} |u|^2 + t^{d+2} |\zeta(l_t) - g_0|^2 \right].$$

1016 Since the bracket is  $\leq 2\Phi(t)$ , we conclude

$$1017 \int_{B(\rho)} |\chi - l_t|^2 + \rho^{d+2} |\zeta(l_t) - g_0|^2 \leq 2C \cdot (\rho/t)^{d+2+2\beta} \Phi(t).$$

Taking the infimum over  $\mathcal{L}$  on the left shows

$$\Phi(\rho) \leq 2C \cdot (\rho/t)^{d+2+2\beta} \Phi(t).$$

Renaming  $2C \rightarrow C$  yields the first term in the asserted inequality.

Step 2. (Control of the lower-order remainder via a trivial competitor) Since  $g_0$  lies in the range of  $\zeta$ , we fix once and for all an  $l_0 \in \mathcal{L}$  with  $\zeta(l_0) = g_0$  and write

$$u := \chi - l_0.$$

Then  $u$  satisfies the homogeneous two-phase system with zero normal-flux-jump across  $\{x_d = 0\}$ . In particular we may apply the refined Caccioppoli–Campanato energy bound of Lemma 12 to  $(u, r)$ : for every  $0 < \rho \leq \frac{1}{2}$

$$\int_{B(\rho)} |\nabla u|^2 + \int_{B(\rho)_-} |\nabla r|^2 \leq C \rho^{d+\alpha} \Psi \left( \frac{1}{2} \right).$$

We now invoke the standard Poincaré inequality on the connected domain  $B(\rho)$ : since  $u$  is continuous across the interface, there exists a constant  $c$  (its average over  $B(\rho)$ ) such that

$$\int_{B(\rho)} |u - c|^2 \leq C \rho^2 \int_{B(\rho)} |\nabla u|^2.$$

But adding or subtracting a constant from  $l_0$  preserves both its piecewise-affine form and its normal-flux-jump, so we may absorb  $c$  into  $l_0$  and still remain in  $\mathcal{L}$ . Hence

$$\Phi(\rho) \leq \int_{B(\rho)} |\chi - l_0|^2 = \int_{B(\rho)} |u - c|^2 \leq C \rho^2 \cdot \left[ \int_{B(\rho)} |\nabla u|^2 \right] \leq C \rho^{d+2+\alpha} \Psi \left( \frac{1}{2} \right).$$

This furnishes the second term in the desired two-scale estimate.

Conclusion. Adding the two estimates of Steps 1 and 2 and re-renaming constants completes the proof: for all  $0 < \rho < t \leq \frac{1}{2}$

$$\Phi(\rho) \leq C \cdot (\rho/t)^{d+2+2\beta} \Phi(t) + C \cdot t^{d+2+\alpha} \Psi \left( \frac{1}{2} \right).$$

□

## Response of AIM

We organize the argument in four main steps:

1. Two-scale excess-decay (Lemma 13). Define the “excess”

$$\Phi(\rho) := \inf_{l \in \mathcal{L}} \left\{ \int_{B(\rho)} |\chi - l|^2 dx + \rho^{d+2} |\zeta(l) - g_0|^2 \right\},$$

where  $\mathcal{L}$  is the finite-dimensional family of piecewise-affine competitors and  $\zeta(l)$  their normal-flux-jump. By combining the refined Caccioppoli–Campanato control (Lemma 12) with the sharp one-sided  $C^{1,\alpha/2}$ -regularity (Lemma 9), one shows:

For every  $\beta \in (\alpha, 1)$  and all  $0 < \rho < t \leq \frac{1}{2}$ ,

$$\Phi(\rho) \leq C \left( \frac{\rho}{t} \right)^{d+2+2\beta} \Phi(t) + C t^{d+2+\alpha} \Psi \left( \frac{1}{2} \right),$$

where  $\Psi \left( \frac{1}{2} \right) := \int_{B(\frac{1}{2})} |\nabla \chi|^2 dx + \int_{B(\frac{1}{2})_-} |\nabla r|^2 dx$  and  $\alpha \in (0, 1)$  is the one-sided exponent from Lemma 9.

1078  
1079  
1080  
1081  
1082  
1083  
1084  
1085  
1086  
1087  
1088  
1089  
1090  
1091  
1092  
1093  
1094  
1095  
1096  
1097  
1098  
1099  
1100  
1101  
1102  
1103  
1104  
1105  
1106  
1107  
1108  
1109  
1110  
1111  
1112  
1113  
1114  
1115  
1116  
1117  
1118  
1119  
1120  
1121  
1122  
1123  
1124  
1125  
1126

2. One-scale Campanato gain (Lemma 10). Fix  $\beta \in (\alpha, 1)$  so close to 1 that the contraction constant

$$\mu := C \cdot \left[ \frac{d+2+\alpha}{d+2+2\beta} \right]^{2\beta-\alpha} < 1.$$

Then Lemma 10 yields a small  $\delta > 0$  (depending only on  $C, \alpha, \beta$ ) and  $C'$  so that for all  $0 < \rho \leq \frac{1}{2}$ ,

$$\Phi(\rho) \leq C' \cdot \rho^{d+2+\alpha+\delta} \left[ \Phi\left(\frac{1}{2}\right) + \Psi\left(\frac{1}{2}\right) \right].$$

In particular,

$$\Phi(\rho) = O\left(\rho^{d+2+\alpha+\delta}\right) \quad \text{as } \rho \rightarrow 0.$$

3. Campanato embedding  $\rightarrow$  one-sided  $C^{1,(\alpha+\delta)/2}$ . The estimate

$$\int_{B(\rho)} |\chi - l_\rho|^2 \leq C'' \rho^{d+2+\alpha+\delta}$$

shows that on each half-ball  $B(\rho) \cap \{\pm x_d \geq 0\}$  the mean oscillation of  $\chi - l_\rho$  is  $O\left(\rho^{1+(\alpha+\delta)/2}\right)$ . By the Campanato characterization of Hölder spaces, one deduces for each sign  $\pm$

$$\chi \in C^{1,(\alpha+\delta)/2}\left(\overline{B_{\frac{1}{2},\pm}}\right),$$

with norm bound  $\leq C \left[ \Phi\left(\frac{1}{2}\right) + \Psi\left(\frac{1}{2}\right) \right]^{\frac{1}{2}}$ .

4. Iterative bootstrap to arbitrary  $\gamma < 1$ . Having now  $C^{1,\gamma_0}$ -control with  $\gamma_0 = (\alpha + \delta)/2$ , whence Campanato  $\rightarrow C^{1,\gamma_1/2}$  etc. Standard iteration shows that the sequence of exponents approaches 1.

Hence for any target  $\gamma < 1$  we may in finitely many steps reach  $\chi \in C^{1,\gamma}\left(\overline{B_{\frac{1}{2},\pm}}\right)$ .

This completes the proof that  $\chi \in C^{1,\gamma}$  on each closed half-ball up to the flat interface, for every  $\gamma < 1$ .

#### B.4 INTERACTIVE ITERATION

This is the content of experiment by the interactive iteration paradigm. After analyzing the output results of AIM, we proposed more precise decomposed problem and clarified the proof directions for AIM. In turn, AIM provided more reasonable proof processes than those generated previously.

##### Prompt to AIM

**Content:** Please try to prove the following lemma.

**Lemma 6.** Suppose  $\Omega$  is a bounded Lipschitz domain in  $\mathbb{R}^d$ . Define the  $\varepsilon$ -neighborhood of the boundary as

$$\Omega_t = \{x \in \Omega : \text{dis}(x, \partial\Omega) < t\}, \quad t > 0.$$

For all  $\psi \in H_0^1(\Omega, \mathbb{R}^d)$ , the following estimate holds:

$$\left| \int_{\Omega} A \nabla w_\varepsilon : \nabla \psi \, dx \right| \leq C \|\nabla \chi\|_{L^2(\Omega)} \left[ \varepsilon \|S_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} + \|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \right] \\ + C \|\nabla \psi\|_{L^2(\Omega_{4\varepsilon})} \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})}.$$

##### Response of AIM

**Theorem 1.** Try to give the reproduction and derivation details of the derivation process below:

1127  
1128 *Lemma 1: Suppose  $\Omega$  is a bounded Lipschitz domain in  $\mathbb{R}^d$ ,  $\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$ ,  $\forall \psi \in$   
1129  $H_0^1(\Omega, \mathbb{R}^d)$*

$$1130 \left| \int_{\Omega} A \nabla w_\varepsilon : \nabla \psi \, dx \right| \leq C \|\nabla \chi\|_{L^2(\Omega)} \left\{ \varepsilon \|S_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} + \|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \right\}$$

$$1131 + C \|\nabla \psi\|_{L^2(\Omega_{4\varepsilon})} \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})}$$

1132  
1133  
1134 *Proof:*

$$1135 A \nabla w_\varepsilon = A \nabla u_\varepsilon - A \nabla u_0 - A \nabla \chi \left( \frac{x}{\varepsilon} \right) S_\varepsilon^2(\nabla u_0) - \varepsilon A \chi(y) \nabla(\eta_\varepsilon S_\varepsilon^2(\nabla u_0))$$

$$1136 = (A \nabla u_\varepsilon - \hat{A} \nabla u_0) + \left[ (\hat{A} - A) \nabla u_0 + (-\hat{A} + A + \nabla \chi \left( \frac{x}{\varepsilon} \right) A + \nabla \chi A \nabla \chi) \eta_\varepsilon S_\varepsilon^2(\nabla u_0) \right]$$

$$1137 - \left[ A + A \nabla \chi \left( \frac{x}{\varepsilon} \right) + \nabla \chi \left( \frac{x}{\varepsilon} \right) A + \nabla \chi A \nabla \chi - \hat{A} \right] \eta_\varepsilon S_\varepsilon^2(\nabla u_0) - \varepsilon A \chi \left( \frac{x}{\varepsilon} \right) \nabla(\eta_\varepsilon S_\varepsilon^2(\nabla u_0))$$

$$1138 = (A \nabla u_\varepsilon - \hat{A} \nabla u_0) + (\hat{A} - A)(\nabla u_0 - \eta_\varepsilon S_\varepsilon^2(\nabla u_0)) + \nabla \chi A \eta_\varepsilon S_\varepsilon^2(\nabla u_0)$$

$$1139 + \nabla \chi A \nabla \chi \eta_\varepsilon S_\varepsilon^2(\nabla u_0) - B \left( \frac{x}{\varepsilon} \right) \eta_\varepsilon S_\varepsilon^2(\nabla u_0) - \varepsilon A \chi \left( \frac{x}{\varepsilon} \right) \nabla(\eta_\varepsilon S_\varepsilon^2(\nabla u_0))$$

1140  
1141  
1142 *Because of the variational form:*

$$1143 \int_{\Omega} A \left( \frac{x}{\varepsilon} \right) \nabla u_\varepsilon : \nabla \psi + \int_{\Omega} p^\varepsilon \cdot \text{div} \, \psi = 0, \quad \forall \psi \in C_0^\infty(\Omega; \mathbb{R}^m)$$

1144  
1145  
1146 *So we can get:*

$$1147 \int_{\Omega} A \nabla w_\varepsilon : \nabla \psi = \int_{\Omega} A \nabla u_\varepsilon : \nabla \psi - \hat{A} \nabla u_0 : \nabla \psi + \int_{D_\varepsilon} p^\varepsilon \cdot \text{div} \, \psi$$

$$1148 + \int_{\Omega} (\hat{A} - A)(\nabla u_0 - \eta_\varepsilon S_\varepsilon^2(\nabla u_0)) + \int_{\Omega} [\nabla \chi A \eta_\varepsilon S_\varepsilon^2(\nabla u_0) + \nabla \chi A \nabla \chi \eta_\varepsilon S_\varepsilon^2(\nabla u_0)] : \nabla \psi$$

$$1149 - \int_{D_\varepsilon} p^\varepsilon \cdot \text{div} \, \psi - \int_{\Omega} B \eta_\varepsilon S_\varepsilon^2(\nabla u_0) : \nabla \psi$$

$$1150 - \int_{\Omega} \varepsilon A \chi \left( \frac{x}{\varepsilon} \right) \nabla(\eta_\varepsilon S_\varepsilon^2(\nabla u_0)) : \nabla \psi$$

1151  
1152  
1153 *And*

$$1154 \int_{\Omega} (\hat{A} - A)(\nabla u_0 - \eta_\varepsilon S_\varepsilon^2(\nabla u_0)) \leq C \int_{\Omega} (1 - \eta_\varepsilon) |\nabla u_0| |\nabla \psi| \, dx$$

$$1155 + C \int_{\Omega} \eta_\varepsilon |\nabla u_0 - S_\varepsilon^2(\nabla u_0)| |\nabla \psi| \, dx$$

$$1156 \leq C \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} \|\nabla \psi\|_{L^2(\Omega_{4\varepsilon})} + C \|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \|\nabla \psi\|_{L^2(\Omega)}$$

1157  
1158 *This is because that:*

$$1159 \|\nabla u_0 - S_\varepsilon^2(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} \leq \|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} + \| - S_\varepsilon(\nabla u_0) - S_\varepsilon^2(\nabla u_0) \|_{L^2(\Omega \setminus \Omega_{3\varepsilon})}$$

$$1160 \leq C \|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})}$$

1176  
1177  
1178  
1179  
1180  
1181  
1182  
1183  
1184  
1185  
1186  
1187  
1188  
1189  
1190  
1191  
1192  
1193  
1194  
1195  
1196  
1197  
1198  
1199  
1200  
1201  
1202  
1203  
1204  
1205  
1206  
1207  
1208  
1209  
1210  
1211  
1212  
1213  
1214  
1215  
1216  
1217  
1218  
1219  
1220  
1221  
1222  
1223  
1224

Moreover, by the regularity of  $\chi \in W^{1,\infty}$ , we have

$$\begin{aligned}
& \int_{\Omega} [\nabla\chi A \eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0) + \nabla\chi A \nabla\chi \eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0)] : \nabla\psi - \int_{D_{\varepsilon}} p^{\varepsilon} \cdot \operatorname{div} \psi \\
& \leq \int_{\Omega} [(\nabla\chi A + \nabla\chi A \nabla\chi)(\eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0) - \nabla u_0)] : \nabla\psi \\
& + \int_{\Omega} (\nabla\chi A + \nabla\chi A \nabla\chi) \nabla u_0 : \nabla\psi - \int_{\omega} p_0 \operatorname{div} \psi + \mathcal{O}(\varepsilon) \\
& = \int_{\Omega} [(\nabla\chi A + \nabla\chi A \nabla\chi)(\eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0) - \nabla u_0)] : \nabla\psi \\
& + \int_{\omega} \delta_{j\beta} r^{i\alpha} \frac{\partial u_0^{\alpha}}{\partial x_i} \frac{\partial \psi^{\beta}}{\partial x_j} - \int_{\omega} \frac{\partial u_0^i}{\partial x_j} r^{ij} \frac{\partial \psi^k}{\partial x_k} + \mathcal{O}(\varepsilon) \\
& = \int_{\Omega} [(\nabla\chi A + \nabla\chi A \nabla\chi)(\eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0) - \nabla u_0)] : \nabla\psi + \mathcal{O}(\varepsilon) \\
& \leq C \|\nabla u_0 - S_{\varepsilon}(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \|\nabla\psi\|_{L^2(\Omega)} + \mathcal{O}(\varepsilon)
\end{aligned}$$

And we also have

$$\begin{aligned}
& \left| \int_{\Omega} B \eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0) : \nabla\psi \right| = \left| \int_{\Omega} b_{ij}^{\alpha\beta} \left(\frac{x}{\varepsilon}\right) S_{\varepsilon}^2 \left(\frac{\partial u_0^{\beta}}{\partial x_j}\right) \left(\frac{\nabla\psi^{\alpha}}{\partial x_i}\right) \eta_{\varepsilon} \right| \\
& = \left| \int_{\Omega} \varepsilon \frac{\partial}{\partial x_k} \left(\phi_{kij}^{\alpha\beta}\right) \left(\frac{x}{\varepsilon}\right) \frac{\partial \psi^{\alpha}}{\partial x_i} S_{\varepsilon}^2 \left(\frac{\partial u_0^{\beta}}{\partial x_j}\right) \eta_{\varepsilon} \right| \\
& \leq C \varepsilon \left[ \int_{\Omega} \eta_{\varepsilon} \left|\phi\left(\frac{x}{\varepsilon}\right)\right| |\nabla\psi| |S_{\varepsilon}^2(\nabla^2 u_0)| dx + \int_{\Omega} |\nabla\eta_{\varepsilon}| \left|\phi\left(\frac{x}{\varepsilon}\right)\right| |\nabla\psi| |S_{\varepsilon}^2(\nabla u_0)| dx \right] \\
& \leq C (\varepsilon \|\nabla\psi\|_{L^2(\Omega)} \|S_{\varepsilon}(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} + \|\nabla\psi\|_{L^2(\Omega_{4\varepsilon})} \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})})
\end{aligned}$$

By the calculations above, we can finally get

$$\begin{aligned}
& \int_{\Omega} A \left(\frac{x}{\varepsilon}\right) \nabla w_{\varepsilon} : \nabla\psi \leq C \|\nabla\psi_{\varepsilon}\|_{L^2(\Omega)} \{ \varepsilon \|S_{\varepsilon}(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} + \|\nabla u_0 - S_{\varepsilon}(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \} \\
& + C \|\nabla w_{\varepsilon}\|_{L^2(\Omega_{4\varepsilon})} \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})}
\end{aligned}$$

**correctness:** True

*Proof.* We decompose  $A(x/\varepsilon)\nabla w_{\varepsilon}$  into five error-terms and then bound each by invoking Lemmas 5–11 above.

1. Algebraic decomposition. By definition

$$w_{\varepsilon} = u_{\varepsilon} - u_0 - \varepsilon\chi(x/\varepsilon)\eta_{\varepsilon}S_{\varepsilon}^2[\nabla u_0].$$

Hence

$$A(x/\varepsilon)\nabla w_{\varepsilon} = A(x/\varepsilon)\nabla u_{\varepsilon} - A(x/\varepsilon)\nabla u_0 - A(x/\varepsilon)\nabla[\varepsilon\chi(x/\varepsilon)\eta_{\varepsilon}S_{\varepsilon}^2\nabla u_0].$$

We expand the last gradient by the product rule:

$$\nabla[\varepsilon\chi(x/\varepsilon)\eta_{\varepsilon}S_{\varepsilon}^2\nabla u_0] = \chi(x/\varepsilon)\eta_{\varepsilon}S_{\varepsilon}^2[\nabla^2 u_0] + \varepsilon(\nabla\chi)(x/\varepsilon)\eta_{\varepsilon}S_{\varepsilon}^2\nabla u_0 + \varepsilon\chi(x/\varepsilon)\nabla(\eta_{\varepsilon}S_{\varepsilon}^2\nabla u_0).$$

In addition, from the cell-problem one shows

$$A(y)(I + \nabla\chi(y)) - \hat{A} =: B(y), \quad \int_Y B(y)dy = 0,$$

and that  $M(y) := \nabla\chi(y)A(y) + \nabla\chi(y)A(y)\nabla\chi(y)$  is bounded, 1-periodic. After a straightforward regrouping one arrives at the identity

$$\begin{aligned} A(x/\varepsilon)\nabla w_\varepsilon &= (A(x/\varepsilon)\nabla u_\varepsilon - \widehat{A}\nabla u_0) + (\widehat{A} - A(x/\varepsilon))(\nabla u_0 - \eta_\varepsilon S_\varepsilon^2 \nabla u_0) \\ &\quad + M(x/\varepsilon)(\eta_\varepsilon S_\varepsilon^2 \nabla u_0 - \nabla u_0) \\ &\quad + B(x/\varepsilon)\eta_\varepsilon S_\varepsilon^2 \nabla u_0 - \varepsilon A(x/\varepsilon)\chi(x/\varepsilon)\nabla(\eta_\varepsilon S_\varepsilon^2 \nabla u_0). \end{aligned}$$

(For full details of the algebra see e.g. [Allaire '92, Appendix A].)

2. Variational cancellation. By the weak form of the  $\varepsilon$ -problem,

$$\int_{\Omega} A(x/\varepsilon)\nabla u_\varepsilon : \nabla\psi dx + \int_{\Omega} p_\varepsilon \operatorname{div}\psi dx = 0,$$

and by the homogenized equation for  $u_0$ ,

$$\int_{\Omega} \widehat{A}\nabla u_0 : \nabla\psi dx = 0.$$

Hence

$$\int_{\Omega} (A(x/\varepsilon)\nabla u_\varepsilon - \widehat{A}\nabla u_0) : \nabla\psi dx = - \int_{\Omega} p_\varepsilon \operatorname{div}\psi dx.$$

But the same pressure-corrector argument shows that the contribution of  $p_\varepsilon$  cancels against the zero-mean of  $M(x/\varepsilon)$ . In fact one checks that

$$\int_{\Omega} (A(x/\varepsilon)\nabla u_\varepsilon - \widehat{A}\nabla u_0) : \nabla\psi + \int_{\Omega} M(x/\varepsilon)(\eta_\varepsilon S_\varepsilon^2 \nabla u_0 - \nabla u_0) : \nabla\psi - \int_{D_\varepsilon} p_\varepsilon \operatorname{div}\psi = 0,$$

so the first grouping and the  $M$ -term combine into a single estimate which is handled by Lemma 10 below. Hence it suffices to bound the four integrals

$$I_1 = \int_{\Omega} (\widehat{A} - A(x/\varepsilon))(\nabla u_0 - \eta_\varepsilon S_\varepsilon^2 \nabla u_0) : \nabla\psi dx,$$

$$I_2 = \int_{\Omega} M(x/\varepsilon)(\eta_\varepsilon S_\varepsilon^2 \nabla u_0 - \nabla u_0) : \nabla\psi dx,$$

$$I_3 = \int_{\Omega} B(x/\varepsilon)\eta_\varepsilon S_\varepsilon^2 \nabla u_0 : \nabla\psi dx,$$

$$I_4 = -\varepsilon \int_{\Omega} A(x/\varepsilon)\chi(x/\varepsilon)\nabla(\eta_\varepsilon S_\varepsilon^2 \nabla u_0) : \nabla\psi dx.$$

We estimate each in turn.

3. Estimate of  $I_1$ . By Lemma 9 (or the periodic-Poincaré estimate plus a trace argument) we have

$$|I_1| \leq C \left[ \varepsilon^{1/2} \|u_0\|_{H^2(\Omega)} + \|\nabla u_0 - S_\varepsilon \nabla u_0\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \right] \|\nabla\psi\|_{L^2(\Omega)}.$$

Since  $\varepsilon^{1/2} \|u_0\|_{H^2(\Omega)} \leq C\varepsilon \|S_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} + C\|\nabla u_0 - S_\varepsilon \nabla u_0\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})}$  by Lemma 5, we obtain exactly the first bracket in the claimed bound.

4. Estimate of  $I_2$ . By Lemma 10 on the "corrector-mismatch" term  $M$  one directly gets

$$|I_2| \leq C \left[ \|\nabla u_0 - S_\varepsilon \nabla u_0\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} + \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} \right] \|\nabla\psi\|_{L^2(\Omega)},$$

which again fits into the first bracket plus a boundary-layer remainder controlled below.

5. Estimate of  $I_3$ . Since  $\int_Y B = 0$  and  $B \in H^s(Y)$ ,  $s > d/2$ , we apply Lemma 7 (fast-oscillation Poincaré):

$$\|B(x/\varepsilon)\varphi\|_{L^2(\Omega)} \leq C \left[ \|\varphi\|_{L^2(\Omega)} + \varepsilon \|\nabla\varphi\|_{L^2(\Omega)} \right].$$

1274  
1275  
1276  
1277  
1278  
1279  
1280  
1281  
1282  
1283  
1284  
1285  
1286  
1287  
1288  
1289  
1290  
1291  
1292  
1293  
1294  
1295  
1296  
1297  
1298  
1299  
1300  
1301  
1302  
1303  
1304  
1305  
1306  
1307  
1308  
1309  
1310  
1311  
1312  
1313  
1314  
1315  
1316  
1317  
1318  
1319  
1320  
1321  
1322

Take  $\varphi = \eta_\varepsilon S_\varepsilon^2 \nabla u_0$ . Then

$$|I_4| = \varepsilon \left| \int_{\Omega} A(x/\varepsilon) \chi(x/\varepsilon) \nabla(\eta_\varepsilon S_\varepsilon^2 \nabla u_0) : \nabla \psi dx \right| \leq \\ C \left[ \varepsilon \|S_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} + \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} \right] \|\nabla \psi\|_{L^2(\Omega)}.$$

But  $\eta_\varepsilon \equiv 1$  on  $\Omega \setminus \Omega_{4\varepsilon}$ , so  $\|\eta_\varepsilon S_\varepsilon^2 \nabla u_0\|_{L^2(\Omega)} \leq \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} + \|S_\varepsilon^2 \nabla u_0 - \nabla u_0\|_{L^2(\Omega \setminus \Omega_{4\varepsilon})} \leq C \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} + C \|\nabla u_0 - S_\varepsilon \nabla u_0\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})}$ , and by Lemma 6,

$$\varepsilon \|\nabla(\eta_\varepsilon S_\varepsilon^2 \nabla u_0)\|_{L^2(\Omega)} \leq C \left[ \varepsilon \|S_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} + \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} \right].$$

Hence

$$|I_3| \leq C \left\{ \varepsilon \|S_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} + \|\nabla u_0 - S_\varepsilon \nabla u_0\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} + \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} \right\} \|\nabla \psi\|_{L^2(\Omega)}.$$

6. Estimate of  $I_4$ . By Lemma 11 on the  $\varepsilon$ -small-term we have

$$|I_4| = \varepsilon \left| \int_{\Omega} A(x/\varepsilon) \chi(x/\varepsilon) \nabla(\eta_\varepsilon S_\varepsilon^2 \nabla u_0) : \nabla \psi dx \right| \leq \\ C \left[ \varepsilon \|S_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} + \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} \right] \|\nabla \psi\|_{L^2(\Omega)}.$$

7. Conclusion. Summing  $I_1 + I_2 + I_3 + I_4$  and observing that all boundary-layer terms are absorbed into the final

$$C \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} \|\nabla \psi\|_{L^2(\Omega_{4\varepsilon})}$$

and the bulk errors combine to

$$C \left\{ \varepsilon \|S_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} + \|\nabla u_0 - S_\varepsilon \nabla u_0\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \right\} \|\nabla \psi\|_{L^2(\Omega)},$$

we recover exactly the claimed estimate. This completes the proof of the lemma.  $\square$

**comment:** None

## B.5 APPLICABILITY BOUNDARY AND EXCLUSION DOMAIN

This is the experimental result of AIM in the reasoning process of multi-symbol operations such as two-dimensional expansion. AIM conducted various explorations, but none of them achieved the expected derivation result. Furthermore, there are not only obvious errors but also numerous processes that are irrelevant to the content we need to derive.

### Prompt to AIM

**Content:** You can use the two-scale expansion to conclude the homogenization. You can perform the expansion on the equation based on this formula  $\nabla := \nabla_x + \frac{1}{\varepsilon} \nabla_y$

**Content:** Let  $u_\varepsilon$  and  $p_\varepsilon$  be the solution of the coupled Lamé-Stokes system. Then

$$u_\varepsilon = u_0(x, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \mathcal{O}(\varepsilon^3); \\ p_\varepsilon = \frac{1}{\varepsilon} p_{-1}(x, y) + p_0(x, y) + \varepsilon p_1(x, y) + \varepsilon^2 p_2(x, y) + \mathcal{O}(\varepsilon^3)$$

And substitute the form of such a solution into the original equation to get the equation of  $u_0, u_1, u_2$ , which means the equations of three scales:  $\mathcal{O}(1), \mathcal{O}(\varepsilon), \mathcal{O}(\varepsilon^2)$ .

## Response of AIM

**Lemma 7.** *Under the hypotheses of Lemmas 5–14 (periodic elastic matrix with Lamé coefficients  $(\lambda, \mu)$ , incompressible Stokes fluid in the  $\varepsilon$ -periodic inclusions  $\omega$ , and two-scale expansions*

$$u_\varepsilon^+(x) = u_0(x) + \varepsilon u_1^+(x, y) + O(\varepsilon^2) \quad \text{in } \Omega \setminus D_\varepsilon,$$

$$u_\varepsilon^-(x) = u_0(x) + \varepsilon u_1^-(x, y) + O(\varepsilon^2),$$

$$p_\varepsilon(x) = \varepsilon^{-1} p_{-1}(x) + p_0^-(x, y) + O(\varepsilon),$$

with continuity  $u_0^+ = u_0^- = u_0$ ,  $Y$ -periodicity in  $y$ , and the usual cell problems for the elastic correctors  $\chi^{pq}(y)$  and Stokes correctors  $(w^k, \pi^k)$ , the macroscopic displacement  $u_0(x)$  and leading pressure  $p_{-1}(x)$  satisfy in  $\Omega$  the homogenized elastic-equilibrium equation

$$-\nabla \cdot [C^* : e(u_0)] + |\omega| \nabla p_{-1} = 0.$$

**correctness:** True

*Proof.* We give a fully rigorous two-scale derivation, correcting the interface-traction argument of the original proof.

1. Two-scale expansions and cell problems a) Elastic (matrix) phase  $Y_f = Y \setminus \bar{\omega}$ . By Lemma 5,  $u_0^+(x, y) = u_0(x)$ . Collecting  $O(\varepsilon^{-1})$  terms in  $\mathcal{L}_{\lambda, \mu} u_\varepsilon^+ = 0$  yields the elasticity-cell problem for  $\{\chi^{pq}\}_{p, q=1}^d$  (Lemma 7). One then shows

$$u_1^+(x, y) = - \sum_{p, q=1}^d \chi^{pq}(y) \partial_{x_q} u_0^p(x).$$

b) Fluid (inclusion) phase  $\omega$ . By Lemma 6,  $p_{-1}(x, y) = p_{-1}(x)$ . Grouping  $O(\varepsilon^{-2})$  in the Stokes system gives the cell-Stokes problem for  $(w^k, \pi^k)$ ,  $k = 1, \dots, d$  (Lemma 9):

$$-\tilde{\mu} \Delta_y w^k + \nabla_y \pi^k = e_k, \quad \operatorname{div}_y w^k = 0 \text{ in } \omega,$$

$$w^k = 0 \text{ on } \partial\omega, \text{ periodic on } \partial Y, \quad \int_\omega \pi^k = 0.$$

Hence

$$u_1^-(x, y) = \sum_{k=1}^d w^k(y) \partial_{x_k} p_{-1}(x),$$

$$p_0^-(x, y) = \sum_{k=1}^d \pi^k(y) \partial_{x_k} p_{-1}(x).$$

2. Homogenized elasticity in the matrix At order  $O(\varepsilon^0)$  in  $\Omega \setminus D_\varepsilon$  one finds

$$-\nabla_y \cdot \sigma_y(u_2^+) - \nabla_x \cdot \sigma_y(u_1^+) - \nabla_y \cdot \sigma_x(u_1^+) - \nabla_x \cdot \sigma_x(u_0) = 0.$$

Integrating over  $Y_f$  and using periodicity plus the cell-problem identities shows

$$\nabla_x \cdot \int_{Y_f} [\sigma_x(u_0) + \sigma_y(u_1^+)] dy = 0,$$

and by Lemma 8

$$\int_{Y_f} [\sigma_x(u_0) + \sigma_y(u_1^+)] dy = C^* : e(u_0).$$

Thus, neglecting fluid, one would obtain

$$-\nabla \cdot [C^* : e(u_0)] = 0.$$

1372 3. Fluid-traction contribution On each inclusion boundary  $\partial\omega_\varepsilon$  the solid traction balances the fluid  
1373 traction. Expand  
1374

$$1375 \quad u_\varepsilon^- = u_0(x) + \varepsilon u_1^-(x, y) + \dots, \quad p_\varepsilon^- = \varepsilon^{-1} p_{-1}(x) + p_0^-(x, y) + \dots,$$

1376 and note  $\nabla \rightarrow \nabla_x + (1/\varepsilon)\nabla_y$ . Then the fluid stress  
1377

$$1378 \quad \sigma^-(u_\varepsilon^-, p_\varepsilon^-) = 2\mu \left[ (1/\varepsilon)D_y(u_1^-) + D_x(u_0) \right] - [\varepsilon^{-1}p_{-1} + p_0^-]I + O(\varepsilon).$$

1379 The  $O(\varepsilon^{-1})$ -term enforces  $\nabla_y p_{-1} = 0$  (so  $p_{-1} = p_{-1}(x)$ ). The  $O(\varepsilon^0)$ -traction on  $\partial\omega$  is  
1380

$$1381 \quad t(y) = [2\mu D_y(u_1^-)(y) - p_0^-(y)I] n(y) + [2\mu D_x(u_0) - p_{-1}I] n(y).$$

1382 Since  $D_x(u_0)$  and  $p_{-1}$  are  $y$ -constant and  $\int_{\partial\omega} n dS = 0$ , the second bracket integrates to zero. Hence  
1383 the net force per cell exerted by the fluid on the solid is  
1384

$$1385 \quad \int_{\partial\omega} t dS = \int_{\partial\omega} [2\mu D_y(u_1^-) - p_0^- I] n dS = \partial_{x_k} p_{-1}(x) \int_{\partial\omega} [2\mu D_y(w^k) - \pi^k I] n dS.$$

1386 By the divergence theorem in  $\omega$ ,  
1387

$$1388 \quad \int_{\partial\omega} [2\mu D_y(w^k) - \pi^k I] n dS = \int_{\omega} \nabla_y \cdot [2\mu D_y(w^k) - \pi^k I] dy$$

$$1389 \quad = \int_{\omega} [\mu \Delta_y w^k - \nabla_y \pi^k] dy = - \int_{\omega} e_k dy = -|\omega| e_k.$$

1390 Hence  
1391

$$1392 \quad \int_{\partial\omega} t dS = -|\omega| \partial_{x_k} p_{-1} e_k = -|\omega| \nabla p_{-1}(x).$$

1393 Dividing by the cell-volume  $\varepsilon^d$  and passing to the limit adds a coupling term  
1394

$$1395 \quad +|\omega| \nabla p_{-1}(x)$$

1396 to the macroscopic equilibrium.  
1397

1398 4. Conclusion Combining the matrix-average (Step 2) with the fluid-traction force (Step 3) yields  
1399 exactly  
1400

$$1401 \quad -\nabla \cdot [C^* : e(u_0)] + |\omega| \nabla p_{-1} = 0$$

1402 in  $\Omega$ . This completes a correct, complete, and rigorous derivation of the homogenized elastic-  
1403 equilibrium equation.  $\square$   
1404

1405 **comment:** None **type:** lemma  
1406  
1407  
1408  
1409  
1410  
1411  
1412  
1413  
1414  
1415  
1416  
1417  
1418  
1419  
1420

## C FULL PROOF FOR THE HOMOGENIZATION PROBLEM

In this section, we present the complete proof of the homogenization problem obtained through human–AI collaboration.

In the following content, we use **blue** to represent content derived manually, **green** to represent content output by AIM and then formatted manually, **orange** to represent content that was manually refined after AIM provided a proof with a high degree of completion and the remaining is the basic setup and information of this problem.

### C.1 INTRODUCTION

Consider  $D \subset \Omega \subset \mathbb{R}^d$  ( $d \geq 2$ ), where  $\Omega$  is elastic material and  $D$  is the high contrast inclusion part.

- $\Omega$  is open bounded with connected  $C^\infty$  boundary  $\partial\Omega$ .
- $D$  is open, has a finite number of components and has a Lipschitz boundary  $\partial D$ .
- $\Omega \setminus D$  is connected with Lipschitz boundary  $\partial\Omega \cup \partial D$ . The connected components of  $D$  are enumerated as  $D_i, i = 1, \dots, N$ ,  $N$  is finite.

And given  $\varepsilon \in (0, 1)$ ,  $D = D_\varepsilon$  is part of an  $\varepsilon$ -periodic array of small inclusions constructed as follows, in several steps.

$Y = (-\frac{1}{2}, \frac{1}{2})^d$  is the unit cell.  $\omega \subset Y$  is a simple connected open subset with connected Lipschitz boundary such that  $\text{dist}(\omega, \partial Y) > 0$ .  $Y_f = Y \setminus \bar{\omega}$  is the model environment in the unit scale.

Given  $\varepsilon > 0$  and  $\mathbf{n} \in \mathbb{Z}^d$ , we denote  $\varepsilon(\mathbf{n} + Y)$  and  $\varepsilon(\mathbf{n} + \omega)$  by  $Y_\varepsilon^{\mathbf{n}}$  and  $\omega_\varepsilon^{\mathbf{n}}$ , respectively. Let  $\Pi_\varepsilon$  be the set of lattice points  $\mathbf{n}$  such that  $\bar{Y}_\varepsilon^{\mathbf{n}}$  be contained in  $\Omega$ , i.e.,

$$\Pi_\varepsilon := \left\{ \mathbf{n} \in \mathbb{Z}^d : \bar{Y}_\varepsilon^{\mathbf{n}} \subset \Omega \right\}, \quad (1)$$

then the inclusions set  $D = D_\varepsilon$  and the background part  $\Omega_\varepsilon$  are defined by

$$D_\varepsilon := \bigcup_{\mathbf{n} \in \Pi_\varepsilon} \omega_\varepsilon^{\mathbf{n}} \quad \Omega_\varepsilon := \Omega \setminus \bar{D}_\varepsilon. \quad (2)$$

A pair of real numbers  $(\lambda, \mu)$  is called admissible and referred to as a Lamé pair, if they satisfy  $\mu > 0$  and  $d\lambda + 2\mu > 0$ . For a Lamé pair  $(\lambda, \mu)$ , the elastostatic system (Lamé system) reads

$$\mathcal{L}_{\lambda, \mu} u := \mu \Delta u + (\lambda + \mu) \nabla \text{div} u, \quad (3)$$

where  $u = (u^1, \dots, u^d)$  represents the displacement field and the divergence of  $u$  is given by  $\text{div} u = \sum_{i=1}^d \frac{\partial u^i}{\partial x_i}$ . The Lamé operator can be written as  $\nabla \cdot \sigma(u)$  where

$$\sigma(u) := \lambda(\nabla \cdot u) \mathbb{I}_d + 2\mu \mathcal{D}(u), \quad (4)$$

$$\mathcal{D}(u) = \frac{1}{2}(\nabla + \nabla^T)u = \frac{1}{2}(\partial_i u^j + \partial_j u^i)_{ij}. \quad (5)$$

The corresponding conormal derivative (boundary traction) at the boundary of a domain  $E$  is

$$\frac{\partial u}{\partial \nu_{(\lambda, \mu)}} \Big|_{\partial E} := \sigma(u)N = \lambda(\text{div} u)N + 2\mu \mathcal{D}(u)N \text{ on } \partial E. \quad (6)$$

We consider the space  $\mathbb{R}$  of rigid motions in  $\mathbb{R}^d$ , defined by

$$\mathbb{R} := \left\{ \mathbf{r} = (r_1, \dots, r_d)^T : \mathcal{D}(\mathbf{r}) = 0 \text{ in } \mathbb{R}^d \right\}.$$

We define  $H_{\mathbb{R}}^{-\frac{1}{2}}(\partial D_\varepsilon)$  as the subspace of  $H^{-\frac{1}{2}}(\partial D_\varepsilon)$  that is orthogonal to  $\mathbb{R}$ , i.e.,

$$H_{\mathbb{R}}^{-\frac{1}{2}}(\partial D_\varepsilon) := \left\{ \phi \in H^{-\frac{1}{2}}(\partial D_\varepsilon) : (\phi, \mathbf{r})_{(H^{\frac{1}{2}}(\partial D_{\varepsilon_i}), H^{-\frac{1}{2}}(\partial D_{\varepsilon_i}))} = 0, \forall \mathbf{r} \in \mathbb{R} \text{ and } 1 \leq i \leq N \right\}. \quad (7)$$

Consider the displacement field  $u_\varepsilon$  satisfying the following transmission system:

$$\begin{cases} \mathcal{L}_{\lambda, \mu} u_\varepsilon = 0 & \text{in } \Omega \setminus \overline{D_\varepsilon}, \\ \mathcal{L}_{\tilde{\lambda}, \tilde{\mu}} u_\varepsilon = 0 & \text{in } D_\varepsilon, \\ u_\varepsilon|_- = u_\varepsilon|_+ \text{ and } \frac{\partial u_\varepsilon}{\partial \nu(\tilde{\lambda}, \tilde{\mu})}|_- = \frac{\partial u_\varepsilon}{\partial \nu(\lambda, \mu)}|_+ & \text{on } \partial D_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu(\lambda, \mu)}|_{\partial \Omega} = g \in H_{\mathbb{R}}^{-\frac{1}{2}}(\partial \Omega) \text{ and } u_\varepsilon|_{\partial \Omega} \in H_{\mathbb{R}}^{\frac{1}{2}}(\partial \Omega). \end{cases} \quad (8)$$

Suppose  $\tilde{\mu}$  fixed, then we arrive at the equations about the **incompressible inclusion limit**. In this case, we need to consider the homogenization problem of the following coupled **Lamé-Stokes system**:

$$\begin{cases} \nabla \cdot (\lambda(\nabla \cdot u_\varepsilon)I + 2\mu D(u_\varepsilon)) = 0 & \text{in } \Omega \setminus \overline{D_\varepsilon}, \\ \nabla \cdot (2\tilde{\mu}D(u_\varepsilon) + p_\varepsilon I) = 0, \nabla \cdot u_\varepsilon & \text{in } D_\varepsilon, \\ (2\tilde{\mu}D(u_\varepsilon) + p_\varepsilon I)N^- - (\lambda(\nabla \cdot u_\varepsilon)I + 2\mu D(u_\varepsilon))N^+ = 0 & \text{on } \partial D_\varepsilon, \\ u_\varepsilon|_+ = u_\varepsilon|_- & \text{on } \partial D_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu(\lambda, \mu)}|_{\partial \Omega} = g \in H_{\mathbb{R}}^{-\frac{1}{2}}(\partial \Omega), u_\varepsilon|_{\partial \Omega} \in H_{\mathbb{R}}^{\frac{1}{2}}(\partial \Omega). \end{cases} \quad (9)$$

## C.2 UNIQUENESS AND EXISTENCE

Define  $V^\varepsilon = \{u \in H^1(\Omega) : u|_{\partial \Omega} \perp \mathbb{R}, \frac{\partial u}{\partial \nu}|_{\partial \Omega} \perp \mathbb{R}\}$ ,  $\|\cdot\|_{V^\varepsilon} = \|\cdot\|_{H^1(\Omega)}$ .

Searching for  $u_\varepsilon \in V^\varepsilon, p_\varepsilon \in L^2(D_\varepsilon)$  such that  $\forall \varphi \in V^\varepsilon, \psi \in L^2(D_\varepsilon)$

$$\begin{aligned} \int_{\partial \Omega} g \cdot \varphi &= \int_{\Omega^\varepsilon} [\lambda(\nabla \cdot u_\varepsilon)I + 2\mu(\frac{1}{2}(\nabla + \nabla^T)u_\varepsilon)] : \nabla \varphi \\ &+ \int_{D_\varepsilon} \operatorname{div} \varphi \cdot p_\varepsilon + \int_{D_\varepsilon} [2\tilde{\mu}(\frac{1}{2}(\nabla + \nabla^T)u_\varepsilon)] : \nabla \varphi, \end{aligned} \quad (10)$$

and

$$0 = \int_{D_\varepsilon} \operatorname{div} u \cdot \psi. \quad (11)$$

Suppose  $a(\cdot, \cdot) : V^\varepsilon \times V^\varepsilon \rightarrow \mathbb{R}$

$$a(u_\varepsilon, \varphi) = \int_{\Omega^\varepsilon} [\lambda(\nabla \cdot u_\varepsilon)I + 2\mu(\frac{1}{2}(\nabla + \nabla^T)u_\varepsilon)] : \nabla \varphi + \int_{D_\varepsilon} [2\tilde{\mu}(\frac{1}{2}(\nabla + \nabla^T)u_\varepsilon)] : \nabla \varphi. \quad (12)$$

$b(\cdot, \cdot) : V^\varepsilon \times L^2(D_\varepsilon) \rightarrow \mathbb{R}$

$$b(u_\varepsilon, \psi) = \int_{D_\varepsilon} \operatorname{div} u_\varepsilon \cdot \psi. \quad (13)$$

So we have the following equation equivalent to the variation form of original equation:

$$\begin{cases} a(u_\varepsilon, \varphi) + b(\varphi, p_\varepsilon) = \int_{\partial \Omega} g \cdot \varphi & \forall \varphi \in V^\varepsilon, \\ b(u_\varepsilon, \psi) = 0 & \forall \psi \in L^2(D_\varepsilon). \end{cases} \quad (14)$$

1519 **Theorem 1** (Babuska-Brezzi Theorem). *Let  $H$  and  $\mathbb{Q}$  be Hilbert spaces, and define bounded bilinear*  
 1520 *forms  $a : H \times H \rightarrow \mathbb{R}$ ,  $b : H \times \mathbb{Q} \rightarrow \mathbb{R}$ . For given  $F \in H'$  and  $G \in \mathbb{Q}'$ , consider finding  $(\sigma, u) \in H \times \mathbb{Q}$*   
 1521 *such that:*

$$1522 \quad a(\sigma, \tau) + b(\tau, u) = F(\tau), \quad \forall \tau \in H,$$

$$1523 \quad b(\sigma, v) = G(v), \quad \forall v \in \mathbb{Q}.$$

1524 *Assume that:*

1525  
 1526 (1) *There exists  $\alpha > 0$  such that*

$$1527 \quad a(\tau, \tau) \geq \alpha \|\tau\|_H^2, \quad \forall \tau \in H.$$

1528  
 1529 (2) *There exists  $\beta > 0$  such that*

$$1530 \quad \inf_{v \in \mathbb{Q}} \sup_{\tau \in H} \frac{b(\tau, v)}{\|\tau\|_H \|v\|_{\mathbb{Q}}} \geq \beta, \quad v \neq 0, \tau \neq 0.$$

1531  
 1532 *Then there exists a unique solution  $(\sigma, u) \in H \times \mathbb{Q}$  satisfying the above equations, and there exists a*  
 1533 *constant  $C = C(\|a\|, \alpha, \beta) > 0$  such that:*

$$1534 \quad \|(\sigma, u)\|_{H \times \mathbb{Q}} \leq C (\|F\|_{H'} + \|G\|_{\mathbb{Q}'}).$$

1535  
 1536 *By this theorem, we verify the conditions to obtain the existence and uniqueness. Suppose  $a_{ij}^{\alpha\beta} =$*   
 1537  *$\lambda \delta_{i\alpha} \delta_{j\beta} + \mu (\delta_{ij} \delta_{\alpha\beta} + \delta_{i\beta} \delta_{j\alpha})$ ,*

$$1541 \quad A \nabla u_\varepsilon \cdot \nabla u_\varepsilon \geq \frac{\min\{\lambda d + 2\mu, 2\mu\}}{4} |(\nabla + \nabla^T) u_\varepsilon|^2, \quad (15)$$

$$1542 \quad \int_{\Omega \setminus D_\varepsilon} A \nabla u_\varepsilon \cdot \nabla u_\varepsilon + \int_{D_\varepsilon} 2\tilde{\mu} \left( \frac{1}{2} (\nabla + \nabla^T) u_\varepsilon \right) : \nabla u_\varepsilon \geq \int_{\Omega \setminus D_\varepsilon} \frac{\min\{\lambda d + 2\mu, 2\mu\}}{4} |(\nabla + \nabla^T) u_\varepsilon|^2$$

$$1543 \quad + 2\tilde{\mu} \int_{D_\varepsilon} \left( \frac{1}{2} (\nabla + \nabla^T) u_\varepsilon \right) : \left( \frac{1}{2} (\nabla + \nabla^T) u_\varepsilon \right) \geq C \|u_\varepsilon\|_{H^1(\Omega)}^2. \quad (16)$$

1544  
 1545  
 1546  
 1547  
 1548  
 1549 **Lemma 1.**

$$1550 \quad \inf_{\psi \in L^2(D_\varepsilon)} \sup_{u_\varepsilon \in V^\varepsilon} \frac{b(u_\varepsilon, \psi)}{\|u_\varepsilon\|_{H^1(\Omega)} \|\psi\|_{L^2(D_\varepsilon)}} \geq \beta. \quad (17)$$

1551  
 1552 Given  $\psi \in L^2(D_\varepsilon)$ ,  $\psi = \sum_{k \in \Pi_\varepsilon} \psi_k$ , here  $\psi_k$  are supported on  $\omega_k^\varepsilon$ . Suppose that  $\tilde{Y}$  is a cubic with  
 1553  $\omega \subset \tilde{Y} \subset Y$ ,  $d(\tilde{Y}, Y) > 0$ ,

$$1554 \quad \hat{\psi}_k = \begin{cases} \psi_k(x) & x \in \omega_k^\varepsilon, \\ -\frac{1}{\varepsilon |\tilde{Y} \setminus \omega|} \int_{\omega_k^\varepsilon} \psi_k(x) dx & x \in \tilde{Y}_k \setminus \omega_k^\varepsilon. \end{cases}$$

1555  
 1556 And we know

$$1557 \quad \|\hat{\psi}\|_{L^2(\tilde{Y}_k)} \leq C_1 \|\psi\|_{L^2(\omega_k^\varepsilon)}$$

1558 , and

$$1559 \quad \int_{\tilde{Y}} \hat{\psi} dx = 0.$$

1560  
 1561 Besides, we can get  $\exists \hat{d}_k \in H_0^1(\tilde{Y}_k)$  such that  $\operatorname{div} \hat{d}_k = \hat{\psi}_k$  in  $\tilde{Y}_k$  with  $\|\nabla \hat{d}_k\|_{L^2(\tilde{Y}_k)} \leq C_2 \|\hat{\psi}\|_{L^2(\tilde{Y}_k)}$ .  
 1562 Note that  $\tilde{d}_k$  is  $\hat{d}_k$ , zero extension to the boundary  $\partial\Omega$  and  $\hat{d} = \sum_{k \in \Pi_\varepsilon} \tilde{d}_k \in V^\varepsilon$ . So we get

$$1563 \quad \|\nabla \hat{d}\|_{L^2(\Omega)} \leq C_2 \|\hat{\psi}\|_{L^2(\Omega)} \leq C_1 \|\psi\|_{L^2(\Omega)}. \quad (18)$$

1568

Therefore,

1569

1570

1571

$$\int_{D_\varepsilon} \psi \cdot \operatorname{div} \hat{d} = \int_{D_\varepsilon} \psi^2 \geq \frac{1}{C_1 C_2} \|\psi\|_{L^2(D_\varepsilon)} \|\nabla \hat{d}\|_{L^2(D_\varepsilon)}. \quad (19)$$

1572

1573

1574

So we verify the inf-sup condition. By the Babuska-Brezzi Theorem, there exists one unique solution  $u_\varepsilon \in V^\varepsilon$ ,  $p_\varepsilon \in L^2(D_\varepsilon)$  to the original equation with  $\|(u_\varepsilon, p_\varepsilon)\|_{V^\varepsilon \times L^2(D_\varepsilon)} \leq C\|g\|_{(V^\varepsilon)^*} \leq C\|g\|_{H^1(\Omega)}$ .

1575

### C.3 UNIQUENESS AND EXISTENCE OF SOLUTIONS AT EACH ORDER OF ASYMPTOTIC EXPANSION

1576

1577

Consider the following equation:  $p, u$  are  $Y$ -periodic

1578

1579

1580

1581

1582

1583

$$\begin{cases} \nabla_y \cdot (\lambda(\nabla_y \cdot u)I + 2\mu(\frac{1}{2}(\nabla + \nabla^T)u)) = F_1 & \text{in } Y \setminus \omega, \\ \nabla_y \cdot (2\tilde{\mu}[\frac{1}{2}(\nabla_y + \nabla_y^T)u] + pI) = F_2, \nabla \cdot u = F_3 & \text{in } \omega, \\ (2\tilde{\mu}[\frac{1}{2}(\nabla_y + \nabla_y^T)u] + p_{-1}I) \cdot N - (\lambda(\nabla \cdot u)I + 2\mu(\frac{1}{2}(\nabla_y + \nabla_y^T)u)) \cdot N = G \cdot N & \text{on } \partial\omega. \\ u|_+ = u|_- & \end{cases} \quad (20)$$

1584

1585

For  $F = F_1(x, \cdot)\mathbf{1}_{y \in Y \setminus \omega} + F_2(x, \cdot)\mathbf{1}_{y \in \omega} \in (H_{\#,0}^1(Y; R^d))^*$ ,  $u \in H_{\#,0}^1(Y; R^d)$ , we define

1586

1587

1588

$$\langle (F, G), u \rangle_{H^*, H} = \int_{Y \setminus \omega} F_1(x, y)u(x, y) dy + \int_{\omega} F_2(x, y)u(x, y) dy + \int_{\partial\omega} G(x, y)u(x, y)N dy. \quad (21)$$

1589

Here,  $F_3(x, \cdot) \in L^2(\omega; R)$ ,  $G(x, \cdot) \in H^{\frac{1}{2}}(\partial\omega; R^d)$ .

1590

And define

1591

1592

1593

$$V = \{u(x, \cdot) \in H_{\#,0}^1(Y; R^d)\}, \quad M = \{p(x, \cdot) \in L^2(\omega; R)\}$$

1594

1595

1596

So the equation has the unique solution  $(u, p) \in V \times M$  if and only if  $\int_{Y \setminus \omega} F_1 + \int_{\omega} F_2 - \int_{\partial\omega} G \cdot N = 0$  and we have  $\|(u, p)\|_{V \times M} \leq C[\|(F_1, F_2, G)\|_{(H_{\#,0}^1(Y; R^d))^*} + \|F_3\|_{L^2(\omega; R)}]$ .

1597

1598

Suppose  $a : H_{\#,0}^1(Y; R^d) \times H_{\#,0}^1(Y; R^d) \rightarrow R$ ,

1599

1600

1601

1602

and  $b : H_{\#,0}^1(Y; R^d) \times L^2(\omega) \rightarrow R$ ,

1603

1604

1605

$$b(u, \psi) = \int_{\omega} \operatorname{div}_y u \cdot \psi. \quad (23)$$

1606

1607

1608

1609

1610

1611

We verify the property of  $a$  and  $b$ :  $\forall u \in H_{\#,0}^1(Y; R^d)$ , and  $a_{ij}^{\alpha\beta} = \lambda\delta_{i\alpha}\delta_{j\beta} + \mu(\delta_{ij}\delta_{\alpha\beta} + \delta_{i\beta}\delta_{j\alpha})$ , there is

$$A\nabla u \cdot \nabla u \geq \frac{\min(\lambda d + 2\mu, 2\mu)}{4} |(\nabla + \nabla^T)u|^2. \quad (24)$$

1612

1613

1614

1615

1616

$$\begin{aligned} & \int_{Y \setminus \omega} A\nabla u \cdot \nabla u + \int_{\omega} 2\tilde{\mu} \left( \frac{1}{2} (\nabla + \nabla^T) u \right) : \nabla u \\ & \geq \int_{Y \setminus \omega} \frac{\min(\lambda d + 2\mu, 2\mu)}{4} |(\nabla + \nabla^T) u|^2 + 2\tilde{\mu} \int_{\omega} \left( \frac{1}{2} (\nabla + \nabla^T) u \right) : \left( \frac{1}{2} (\nabla + \nabla^T) u \right) \\ & \geq C\|u\|_{H^1(Y)}^2. \end{aligned} \quad (25)$$

1617 So we verify this property:

$$1618 \inf_{\psi \in L^2(\omega)} \sup_{u \in H_{\#,0}^1(Y;R^d)} \frac{b(u, \psi)}{\|u\|_{H_{\#,0}^1(Y;R^d)} \|\psi\|_{L^2_\#(Y)}} \geq \beta. \quad (26)$$

1621 Given  $\psi \in L^2_\#(\omega)$ , we consider

$$1622 \hat{\psi}(x) = \begin{cases} \psi(x) & x \in \omega, \\ -\frac{1}{|Y \setminus \omega|} \int_\omega \psi(x) dx & x \in Y \setminus \omega. \end{cases}$$

1623 This extension satisfy  $\|\hat{\psi}\|_{L^2(Y)} \leq C_1 \|\psi\|_{L^2(\omega)}$ ,  $\hat{\psi} \in H_{\#,0}^1(Y;R^d)$  and  $\int_Y \hat{\psi} dx = 0$ .

1624 Besides, we can get  $\exists \hat{d} \in H_{\#,0}^1(Y;R^d)$  such that  $\operatorname{div} \hat{d} = \hat{\psi}$  in  $Y$  with  $\|\nabla \hat{d}\|_{L^2(Y)} \leq C_2 \|\hat{\psi}\|_{L^2(Y)}$ .

1625 So we get

$$1626 \|\nabla \hat{d}\|_{L^2(Y)} \leq C_2 \|\hat{\psi}\|_{L^2(Y)} \leq C_1 C_2 \|\psi\|_{L^2(\omega)}$$

1627 . Therefore,

$$1628 \int_\omega \psi \cdot \operatorname{div} \hat{d} = \int_\omega \psi^2 \geq \frac{1}{C_1 C_2} \|\psi\|_{L^2(\omega)} \|\nabla \hat{d}\|_{L^2(Y)}$$

1629 Finally, we verify that the necessary conditions are sufficient. By divergence theorem,

$$1630 \int_{Y \setminus \omega} F_1 = \int_{Y \setminus \omega} \nabla_y \cdot (\lambda(\nabla_y \cdot u)I + 2\mu(\frac{1}{2}(\nabla + \nabla^T)u)) \quad (27)$$

$$1631 = - \int_{\partial\omega} (\lambda(\nabla_y \cdot u)I + 2\mu(\frac{1}{2}(\nabla + \nabla^T)u)N).$$

$$1632 \int_\omega F_2 = \int_\omega \nabla_y \cdot (2\tilde{\mu}[\frac{1}{2}(\nabla_y + \nabla_y^T)u] + pI) \quad (28)$$

$$1633 = \int_{\partial\omega} (2\tilde{\mu}[\frac{1}{2}(\nabla_y + \nabla_y^T)u] + pI)N.$$

1634 Therefore we get

$$1635 \int_{Y \setminus \omega} F_1 + \int_\omega F_2 - \int_{\partial\omega} G \cdot N = 0. \quad (29)$$

1636 On the other hand, we try to search for  $u \in V, p \in M$  such that for  $\forall \varphi \in V$

$$1637 \int_{Y \setminus \omega} F_1 + \int_\omega F_2 = \int_{Y \setminus \omega} (\lambda(\nabla_y \cdot u)I + 2\mu(\frac{1}{2}(\nabla + \nabla^T)u)) : \nabla_y \varphi. \quad (30)$$

$$1638 \int_{Y \setminus \omega} F_1 + \int_\omega F_2 = - \int_{Y \setminus \omega} (\lambda(\nabla_y \cdot u)I + 2\mu(\frac{1}{2}(\nabla + \nabla^T)u)) : \nabla_y \varphi \quad (31)$$

$$1639 - \int_\omega (2\tilde{\mu}[\frac{1}{2}(\nabla_y + \nabla_y^T)u] + pI) : \nabla_y \varphi + \int_{\partial\omega} G \cdot N \varphi.$$

1640 This is correct since the analysis of Babuska-Brezzi Theorem: the following two equation are well-posedness in  $V \times M$

$$1641 0 = \int_{Y \setminus \omega} (\lambda(\nabla_y \cdot u)I + 2\mu(\frac{1}{2}(\nabla + \nabla^T)u)) : \nabla_y \varphi + \int_\omega (2\tilde{\mu}[\frac{1}{2}(\nabla_y + \nabla_y^T)u] + pI) : \nabla_y \varphi \quad \forall \varphi \in V, \quad (32)$$

1642 and

$$1643 \int_\omega F_3 \cdot \psi = \int_\omega (\nabla_y \cdot u) \psi \quad \forall \psi \in M. \quad (33)$$

1666 C.4 HOMOGENIZATION EQUATION  
1667

1668 For  $(u_0, p_{-1})$ :  
1669

$$1670 \begin{cases} \nabla_y \cdot (\lambda(\nabla_y \cdot u_0 + 2\mu D_y(u_0))) = 0 \\ 1671 \nabla_y \cdot (2\tilde{\mu}D_y(u_0) + p_{-1}I) = 0, \nabla_y \cdot u_0 = 0 \\ 1672 (2\tilde{\mu}D_y(u_0) + p_{-1}I)N^- - (\lambda(\nabla_y \cdot u_0)I + 2\mu D_y(u_0))N^+ = 0 \\ 1673 u_0|_+ = u_0|_- \end{cases} \quad (34)$$

1674 From the derivations in Sec. C.2, we know  $u_0(x, y) = u_0(x)$ ,  $p_{-1}(x, y) = p_{-1}(x) = 0$ .

1676 For  $(u_1, p_0)$ :  
1677

$$1678 \begin{cases} \nabla_y \cdot (\lambda \nabla_y \cdot u_1 + 2\mu D_y(u_1)) = 0 \\ 1679 \nabla_y \cdot (2\tilde{\mu}D_y(u_1) + p_0I) = 0, \nabla_y \cdot u_1 = -\nabla_x \cdot u_0 \\ 1680 (2\tilde{\mu}D_y(u_1) + p_0I)N^- - (\lambda(\nabla_y \cdot u_1)I + 2\mu D_y(u_1))N^+ \\ 1681 = -2\tilde{\mu}D_x(u_0)N^- + (\lambda \nabla_x \cdot u_0 I + 2\mu D_x(u_0))N^+ \\ 1682 u_1|_+ = u_1|_- \end{cases} \quad (35)$$

1683 C.5 CELL PROBLEM  
1684

1685 Suppose  $\chi^{ij}$  and  $r^{ij}$  satisfy the following equation:  
1686

$$1687 \begin{cases} \nabla_y \cdot (\lambda \nabla_y \cdot \chi^{ij} + 2\mu D_y \chi^{ij}) = 0 & \text{in } Y \setminus \omega, \\ 1688 \nabla_y \cdot (2\tilde{\mu}D_y \chi^{ij} + r^{ij}I) = 0, \nabla_y \cdot \chi^{ij} = -\delta_{ij} & \text{in } \omega, \\ 1689 (2\tilde{\mu}D_y(\chi^{ij}) + r^{ij}I)N^- - (\lambda(\nabla_y \cdot \chi^{ij})I + 2\mu D_y(\chi^{ij}))N^+ \\ 1690 = -\tilde{\mu}(E_{ij} + E_{ji})N^- + \mu(E_{ij} + E_{ji})N^+ + \lambda IN \delta_{ij}|_+ \\ 1691 \chi^{ij}|_+ = \chi^{ij}|_- & \text{on } \partial\omega. \end{cases} \quad (36)$$

1693 From the definition above, we have  $\chi^{ij} = \chi^{ji}$ ,  $r^{ij} = r^{ji}$ ,  
1694

$$1695 u_1 = (D_x u_0)^{ij} \cdot \chi^{ij}, p_0 = (D_x u_0)^{ij} r^{ij}$$

1697 .  
1698 So for  $(u_2, p_1)$ , we have:  
1699

$$1700 \begin{cases} \nabla_y \cdot (\lambda \nabla_y \cdot u_2 + 2\mu D_y u_2) = -\nabla_x \cdot [(\lambda(\nabla_x u_0)I + 2\mu D_x) + (\lambda(\nabla_y \cdot u_1)I + 2\mu D_y u_1)] \\ 1701 \quad \quad \quad - \nabla_y \cdot (\lambda(\nabla_x \cdot u_1)I + 2\mu D_x u_1) \\ 1702 \nabla_y \cdot (2\tilde{\mu}D_y u_2 + p_1I) = -\nabla_x \cdot (2\tilde{\mu}D_x u_0 + p_0I) - \nabla_x \cdot (2\tilde{\mu}(D_y u_1)) - \nabla_y \cdot (2\tilde{\mu}(D_x u_1)) \\ 1703 \nabla \cdot u_2 = -\nabla_x \cdot u_1 \\ 1704 u_2|_- = u_2|_+ \\ 1705 (p_1 + 2\tilde{\mu}(D_y u_2))N^- - (\lambda(\nabla_y \cdot u_2) + 2\mu(D_y u_2))N^+ \\ 1706 = -(2\tilde{\mu}D_x(u_1))N^- + (\lambda(\nabla_x \cdot u_1) + 2\mu(D_x u_1))N^+ \end{cases} \quad (37)$$

1708 By divergence theorem, we can get  
1709

$$1710 \int_{\partial\omega} (\lambda(\nabla_x \cdot u_1) + 2\mu(D_x u_1))N^+ = - \int_{Y \setminus \omega} \nabla_y \cdot (\lambda(\nabla_x \cdot u_1) + 2\mu(D_x u_1)). \quad (38)$$

$$1712 \int_{\partial\omega} (2\tilde{\mu}D_x(u_1))N^- = \int_{\omega} \nabla_y \cdot (2\tilde{\mu}D_x(u_1)). \quad (39)$$

1714 Combining the solvability conditions, we have

$$\begin{aligned}
& |Y \setminus \omega| \nabla_x \cdot [\lambda(\nabla_x \cdot u_0)I + 2\mu D_x(u_0)] + \int_{Y \setminus \omega} \lambda(\nabla_x(\nabla_y \cdot u_1)) + 2\mu(\nabla_x \cdot D_y(u_1)) \, dy \\
& + |\omega|(2\tilde{\mu}\nabla_x \cdot D_x u_0) + \int_{\omega} (2\tilde{\mu}(\nabla_x \cdot D_y u_1) + \nabla_x p_0) \, dy = 0.
\end{aligned} \tag{40}$$

Substituting this equation  $u_1 = (D_x u_0)^{ij} \cdot \chi^{ij}$ ,  $p_0 = (D_x u_0)^{ij} r^{ij}$ , we finally get

$$\begin{aligned}
& |Y \setminus \omega| \nabla_x \cdot [\lambda(\nabla_x \cdot u_0)I + 2\mu D_x(u_0)] + \int_{Y \setminus \omega} [\lambda(\nabla_y \cdot \chi^{ij})I + 2\mu D_y(\chi^{ij})](\nabla_x(D_x u_0)^{ij}) \, dy \\
& + |\omega|(2\tilde{\mu}\nabla_x \cdot D_x u_0) + \int_{\omega} (2\tilde{\mu}(D_y \chi^{ij}) + r^{ij}I)\nabla_x(D_x u_0)^{ij} \, dy = 0.
\end{aligned} \tag{41}$$

We can rewrite this equation as the form  $-\frac{\partial}{\partial x_i} \hat{a}_{ij}^{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} = 0$ :

$$\begin{aligned}
\hat{a}_{ij}^{\alpha\beta} &= |Y \setminus \omega| (\lambda \delta_{i\alpha} \delta_{j\beta} + \mu(\delta_{i\beta} \delta_{j\alpha} + \delta_{ij} \delta_{\alpha\beta})) + \int_{Y \setminus \omega} [\lambda \nabla \cdot \chi^{j\beta} \delta_{i\alpha} + \mu(\frac{\partial \chi_\alpha^{j\beta}}{\partial y_i} + \frac{\partial \chi_i^{j\beta}}{\partial y_\alpha})] \, dy \\
& + |\omega| \tilde{\mu}(\delta_{ij} \delta_{\alpha\beta} + \delta_{i\beta} \delta_{j\alpha}) + \int_{\omega} \tilde{\mu}(\frac{\partial \chi_\alpha^{j\beta}}{\partial y_i} + \frac{\partial \chi_i^{j\beta}}{\partial y_\alpha}) + r^{j\beta} \delta_{i\alpha} \, dy.
\end{aligned} \tag{42}$$

### C.5.1 SYMMETRY

We first prove the symmetry of  $\hat{a}_{ij}^{\alpha\beta}$ .

Define:  $a(\varphi, \psi) = \int_{Y \setminus \omega} \nabla_y \varphi : [\lambda(\nabla_y \cdot \psi)I + 2\mu D(\psi)] + \int_{\omega} 2\tilde{\mu} D\psi : D\psi = \int_{Y \setminus \omega} \lambda(\nabla_y \cdot \varphi)(\nabla_y \cdot \psi) + \mu D(\psi) : D(\varphi) + \int_{\omega} 2\tilde{\mu} D\psi : D\varphi$ .

Claim  $\hat{a}_{ij}^{\alpha\beta} = a(p^{i\alpha} + \chi^{i\alpha}, p^{j\beta} + \chi^{j\beta})$ ,

$$a(p^{i\alpha}, p^{j\beta}) = \int_{Y \setminus \omega} \lambda \delta_{i\alpha} \delta_{j\beta} + \mu(\delta_{i\beta} \delta_{j\alpha} + \delta_{ij} \delta_{\alpha\beta}) + \int_{\omega} \tilde{\mu}(\delta_{ij} \delta_{\alpha\beta} + \delta_{i\beta} \delta_{j\alpha}), \tag{43}$$

$$\begin{aligned}
a(p^{i\alpha}, \chi^{j\beta}) &= \int_{Y \setminus \omega} \lambda \delta_{i\alpha} (\nabla \cdot \chi^{j\beta}) + \mu \left( \frac{e_{i\alpha} + e_{\alpha i}}{2} \right) : (\nabla \chi^{j\beta} + \nabla^T \chi^{\beta j}) \\
& + \int_{\omega} \tilde{\mu} \left( \frac{e_{i\alpha} + e_{\alpha i}}{2} \right) : (\nabla \chi^{j\beta} + \nabla^T \chi^{\beta j}) \\
& = \int_{\partial\omega} -\lambda \delta_{i\alpha} \chi^{j\beta} N - \mu(e_{i\alpha} + e_{\alpha i}) \chi^{j\beta} N + \int_{\partial\omega} \tilde{\mu}(e_{i\alpha} + e_{\alpha i}) \chi^{j\beta} N,
\end{aligned} \tag{44}$$

$$\begin{aligned}
a(\chi^{i\alpha}, \chi^{j\beta}) &= \int_{Y \setminus \omega} \lambda(\nabla \cdot \chi^{i\alpha})(\nabla \cdot \chi^{j\beta}) + \mu(\nabla_y \chi^{i\alpha}) : (\nabla \chi^{j\beta} + \nabla^T \chi^{j\beta}) \\
& + \int_{\omega} \tilde{\mu}(\nabla_y \chi^{i\alpha}) : (\nabla \chi^{j\beta} + \nabla^T \chi^{j\beta}) \\
& = \int_{\partial\omega} -[\lambda \chi^{i\alpha} (\nabla \cdot \chi^{j\beta}) + 2\mu \chi^{i\alpha} D(\chi^{j\beta})] N + \int_{\partial\omega} 2\tilde{\mu} \chi^{i\alpha} (D\chi^{j\beta}) N \\
& + \int_{\partial\omega} \chi^{i\alpha} r^{j\beta} N + \int_{\omega} \delta_{i\alpha} r^{j\beta}.
\end{aligned} \tag{45}$$

Observing that  $a(p^{i\alpha}, p^{j\beta} + \chi^{j\beta}) = \hat{a}_{ij}^{\alpha\beta} - \int_{\omega} \delta_{i\alpha} r^{j\beta} \, dy$  and  $a(\chi^{i\alpha}, p^{j\beta} + \chi^{j\beta}) = \int_{\omega} \delta_{i\alpha} r^{j\beta} \, dy$ , we have  $\hat{a}_{ij}^{\alpha\beta} = a(p^{i\alpha} + \chi^{i\alpha}, p^{j\beta} + \chi^{j\beta})$  and  $\hat{a}_{ij}^{\alpha\beta} = \hat{a}_{ji}^{\alpha\beta} = \hat{a}_{ji}^{\beta\alpha}$ .

## 1764 C.5.2 ELLIPTIC

1765 Let  $\varphi = \phi + \varepsilon(D_x\phi)^{ij}\chi^{ij}(\frac{x}{\varepsilon})$ ,  $\phi \in C_0^\infty(\Omega; R^d)$ 

$$\begin{aligned}
1766 & \int_{\Omega} a(\varphi, \varphi) = - \int_{\Omega} \int_{Y \setminus \omega} (\nabla_x \phi + (D_x \phi)^{ij} \nabla_y \chi^{ij}) : [\lambda(\nabla_x \cdot \phi + (D_x \phi)^{ij} \nabla_y \cdot \chi^{ij}) I \\
1767 & \quad + 2\mu(D_x \phi + (D_x \phi)^{ij} D_y(\chi^{ij}))] \\
1768 & \quad - \int_{\Omega} 2\tilde{\mu} \int_{\omega} (\nabla_x \phi + (D_x \phi)^{ij} \nabla_y \chi^{ij}) : (D_x \phi + (D_x \phi)^{ij} (D_y \chi^{ij})) \\
1769 & = \int_{\Omega} \int_{Y \setminus \omega} \nabla_x \cdot (\lambda(\nabla \cdot \phi) I + 2\mu D_x \phi) + (\lambda(\nabla_y \cdot \chi^{ij}) I + 2\mu(D_y \chi^{ij})) (\nabla_x (D_x \phi)^{ij}) dy \\
1770 & \quad + \int_{\omega} 2\tilde{\mu} \nabla_x \cdot D_x \phi + (2\tilde{\mu}(D_y \chi^{ij}) + r^{ij} I) \nabla_x (D_x \phi)^{ij} dy \\
1771 & = \int_{R^d} \hat{A} \nabla \phi : \nabla \phi. \\
1772 & \\
1773 & \\
1774 & \\
1775 & \\
1776 & \\
1777 & \\
1778 & \\
1779 & \\
1780 & \tag{46}
\end{aligned}$$

1781 So we have

$$\begin{aligned}
1782 & \frac{\min\{\lambda d + 2\mu, 2\mu, \tilde{\mu}\}}{2} \int_{R^d} |\nabla \phi|^2 dx \leq \liminf_{\varepsilon \rightarrow 0} \frac{\min\{\lambda d + 2\mu, 2\mu, \tilde{\mu}\}}{2} \int_{R^d} |\nabla \varphi|^2 dx \leq \int_{R^d} \hat{A} \nabla \phi : \nabla \phi dx. \\
1783 & \\
1784 & \tag{47}
\end{aligned}$$

1785 Finally we can get

$$\begin{aligned}
1786 & \frac{\min\{\lambda d + 2\mu, 2\mu, \tilde{\mu}\}}{2} |\xi|^2 |\eta|^2 \leq \hat{a}_{ij}^{\alpha\beta} \xi_i \xi_j \eta^\alpha \eta^\beta \\
1787 & \\
1788 & . \\
1789 &
\end{aligned}$$

## 1790 C.6 REGULARITY

1791 For each  $i, j = 1, \dots, d$ , suppose  $p^{ij} = \frac{1}{2}(y_j e_i + y_i e_j)$ ,  $\chi^{ij}(y)$  and  $r^{ij}(y)$  satisfy the following equation:

$$\begin{cases}
1792 & \nabla \cdot [\lambda \nabla_y \cdot \chi^{ij} I + 2\mu D_y u] = 0 & \text{in } Y \setminus \omega, \\
1793 & \nabla \cdot [r^{ij} I + 2\tilde{\mu} D_y \chi^{ij}] = 0 & \text{in } \omega, \\
1794 & \nabla_y \cdot \chi^{ij} = -\delta_{ij} & \text{in } \omega, \\
1795 & \chi^{ij}|_+ = \chi^{ij}|_-, \\
1796 & [r^{ij} I + 2\tilde{\mu} D_y \chi^{ij}] N|_- - [\lambda \nabla_y \cdot \chi^{ij} I + 2\mu D_y \chi^{ij}] N|_+ \\
1797 & \quad = -\tilde{\mu}(E_{ij} + E_{ji}) N|_- + \mu(E_{ij} + E_{ji}) N|_+ + \lambda I N \delta_{ij}|_+ & \text{on } \partial\omega.
\end{cases}$$

1801 In  $Y \setminus \omega$  and  $\omega$ , by interior estimation for  $V \subset\subset Y \setminus \omega$ ,  $W \subset\subset \omega$ , we have  $\|\chi\|_{H^k(V)} < \infty$ ,  $\|\chi\|_{H^k(W)} < \infty$ . So we can consider the following boundary regularity problem on  $\partial\omega$ .

1802 Consider:

$$\begin{cases}
1803 & \lambda \Delta_y (\chi^{ij} + p^{ij}) + (\lambda + \mu) \nabla_y (\nabla_y \cdot (\chi^{ij} + p^{ij})) = 0 & \text{in } Y \setminus \omega, \\
1804 & \tilde{\mu} \Delta_y (\chi^{ij} + p^{ij}) + \nabla_y r^{ij} = 0 & \text{in } \omega, \\
1805 & \nabla_y \cdot (\chi^{ij} + p^{ij}) = 0 & \text{in } \omega, \\
1806 & \chi^{ij}|_+ = \chi^{ij}|_-, \\
1807 & [r^{ij} I + 2\tilde{\mu} D_y (\chi^{ij} + p^{ij})] N|_- - [\lambda \nabla_y \cdot (\chi^{ij} + p^{ij}) I + 2\mu D_y (\chi^{ij} + p^{ij})] N|_+ = 0 & \text{on } \partial\omega.
\end{cases}$$

1811 This is equivalent to the following problem:

1813 Suppose that there is a sphere  $Q_r = (-r, r)^d$  with radius  $r$  centered at the origin in  $\mathbb{R}^d$ .

1814 We define

$$1815 \quad Q_r^+ = \{(x', x_d) \in Q_r : x_d > \psi(x')\}, \quad Q_r^- = \{(x', x_d) \in Q_r : x_d < \psi(x')\},$$

$$1816 \quad I_r = \{(x', x_d) \in Q_r : x_d = \psi(x')\}.$$

1817 Here  $\psi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  is a  $C^\infty$  function and  $\psi(0) = 0$ ,

$$1818 \quad \begin{cases} \nabla_y \cdot [\lambda \nabla_y \cdot \chi^{ij} I + 2\mu D_y \chi^{ij}] = 0 & \text{in } Q_r^+, \\ \nabla \cdot [r^{ij} I + 2\tilde{\mu} D_y \chi^{ij}] = 0 & \text{in } Q_r^-, \\ \nabla \cdot \chi^{ij} = 0 & \text{in } Q_r^-, \\ \chi^{ij}|_+ = \chi^{ij}|_- & \\ [r^{ij} I + 2\tilde{\mu} D_y \chi^{ij}] N|_- - [\lambda \nabla_y \cdot \chi^{ij} I + 2\mu D_y \chi^{ij}] N|_+ = 0 & \text{on } I_r. \end{cases}$$

1819  $\chi^{ij}, r^{ij}$  are the weak solution in  $H^1(Q_r; \mathbb{R}^d)$  and  $L^2(Q_r^-; \mathbb{R})$ .

### 1820 C.6.1 A BASIC $C^{1,\alpha}$ ESTIMATE

1821 Suppose  $\Omega_\pm = \mathbb{R}_\pm^d$ ,  $S = \{x_d = 0\}$ ,  $B_+ = \{x \in B(1) : x_d > 0\}$ , and  $B_- = \{x \in B(1) : x_d < 0\}$ ,

1822 where  $B(1) = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ .

1823 Consider the following equation: for all  $V \in H_0^1(B(1); \mathbb{R}^d)$ ,

$$1824 \quad \begin{cases} (\nabla V : A_1 \nabla \tilde{\chi})_{B_+} + (\nabla V : A_2 \nabla \tilde{\chi})_{B_-} + (\tilde{r}, \nabla \cdot (aV))_{B_-} = 0, \\ \nabla \cdot (a\tilde{\chi}) = 0. \end{cases}$$

1825 where  $\tilde{\chi} = D^\alpha \chi$ ,  $\tilde{r} = D^\alpha r$  with  $|\alpha| \geq 1$ ,  $A_1, A_2$  are constant tensors, and  $a$  is a constant matrix.

1826 We let  $V = \eta^2 \tilde{\chi}$  and  $\eta$  be smooth with  $\eta = 0$  in  $B(1)^c$  and  $\eta = 1$  in  $B(\frac{1}{2})$ . So we can get

$$1827 \quad \begin{aligned} & \eta^2 (\nabla \tilde{\chi} : A_1 \nabla \tilde{\chi})_{B_+} + \eta^2 (\nabla \tilde{\chi} : A_2 \nabla \tilde{\chi})_{B_-} + (\tilde{\chi} (\nabla \eta^2)^T : A_1 \nabla \tilde{\chi})_{B_+} \\ & + (\tilde{\chi} (\nabla \eta^2)^T : A_2 \nabla \tilde{\chi})_{B_-} + (\tilde{r}, \nabla \cdot (a\tilde{\chi}) \eta^2)_{B_-} + (\tilde{r}, (\nabla \eta^2)^T \cdot (a\tilde{\chi}))_{B_-} = 0. \end{aligned}$$

1828 Therefore,

$$1829 \quad \|\eta \nabla \tilde{\chi}\|_{L^2(B(1))}^2 \leq C \|\eta \nabla \tilde{\chi}\|_{L^2(B(1))} \|\tilde{\chi}\|_{L^2(B(1))} + C \|\eta \tilde{r}\|_{L^2(B(1))} \|\tilde{\chi}\|_{L^2(B(1))}. \quad (48)$$

1830 Besides, we also have  $\|\eta \tilde{r}\|_{L^2(B(1))} \leq C \sum_{\beta=|\alpha|+1} \|\eta D^\beta \chi\|_{L^2(B(1))} = m_{|\alpha|+1}$ , and  $\sum_{\beta=|\alpha|} \|D^\beta \chi\|_{L^2(B(1))} = n_{|\alpha|}$ .

1831 So we have  $m_{|\alpha|+1}^2 \leq C m_{|\alpha|+1} n_{|\alpha|}$ . That is

$$1832 \quad \|D^{|\alpha|+1} \chi\|_{L^2(B(\frac{1}{2}))} \leq C \|D^{|\alpha|} \chi\|_{L^2(B(1))}. \quad (49)$$

1833 Similarly,

$$1834 \quad \|D^{|\alpha|} r\|_{L^2(B(\frac{1}{2}))} \leq C \|D^{|\alpha|-1} r\|_{L^2(B(\frac{1}{2}))}. \quad (50)$$

### 1835 C.6.2 SCHAUDER THEORY

1836 Since  $\frac{\partial^2 \chi}{\partial y_d^2}$  is the linear combination of  $\frac{\partial^2 \chi}{\partial y_i \partial y_j}$ ,  $\frac{\partial \chi}{\partial y_j}$ ,  $\frac{\partial r}{\partial y_j}$ ,  $r$ , the inequalities (Eq. 49 and Eq. 50) above are correct for any  $|\alpha_n| = |\alpha|$ . So we have proved the Caccioppoli inequality.

1862 **Lemma 2.** (Caccioppoli Inequality) For the weak solutions  $(\chi, r)$ ,  $\forall k \geq 1$

$$1863 \sum_{\pm} \|\chi\|_{H^k(B(\frac{1}{2}, \pm))} \leq C \|\chi\|_{L^2(B(1))}, \quad (51)$$

$$1864 \quad \quad \quad \|r\|_{H^k(B(\frac{1}{2})_-)} \leq C \|r\|_{L^2(B(\frac{1}{2})_-)}. \quad (52)$$

1865  
1866  
1867  
1868  
1869 **Lemma 3.** Suppose that  $M$  is the constant matrix in  $\mathbb{R}^{d \times d}$ , the following are equivalent:

$$1870 \begin{cases} \forall y \in \{y_d = 0\}, M_+ x = M_- x, \\ \exists c \in \mathbb{R}^d \text{ s.t. } M_+ - M_- = C e_d^T, \\ (I - e_d^T e_d) M_+ = (I - e_d^T e_d) M_-. \end{cases} \quad (53)$$

1871  
1872  
1873  
1874  
1875 **Definition 1.** Let  $A_1, A_2$  be constant tensors and  $a$  be a constant matrix. Suppose  $M_+$  and  $M_-$  satisfy  
1876 Lemma 3 above and  $\nabla \cdot (a M_- y) = 0$  in  $B(t)_-$ . Define

$$1877 l(y) = M_+ y \mathbf{1}_{y_d \geq 0} + M_- y \mathbf{1}_{y_d \leq 0} + C, \quad q(y) = r(0),$$

1878  
1879 where  $\mathbf{1}$  denotes the indicator function. We call  $l, q$  the piecewise linear solutions of the following  
1880 equations:

$$1881 \begin{cases} \nabla \cdot (A_1 \nabla l) = 0 & \text{in } \mathbb{R}_+^d, \\ \nabla \cdot (A_2 \nabla l) + \nabla \cdot (a^T q) = 0 & \text{in } \mathbb{R}_-^d, \\ \nabla \cdot (a l) = 0 & \text{in } \mathbb{R}_-^d, \\ l|_+ = l|_- & \text{on } \{x_d = 0\}, \\ \left. \frac{\partial l}{\partial \nu} \right|_+ - \left. \frac{\partial l}{\partial \nu} \right|_- = (A_1 M_+) e_d - (A_2 M_- + a^T r(0)) e_d & \text{on } \{x_d = 0\}. \end{cases}$$

1882  
1883  
1884  
1885  
1886  
1887  
1888  
1889 where  $e_d$  is the standard basis vector in  $\mathbb{R}^d$ .

1890 Let  $\mathcal{L}$  be the space of all piecewise linear solutions of the above equations. For any  $(l, q) \in \mathcal{L}$ , define

$$1891 \zeta(l, q) = \left( \frac{\partial l}{\partial \nu} \right)_+ - \left( \frac{\partial l}{\partial \nu} \right)_-.$$

1892  
1893  
1894  
1895 **Lemma 4.** Let  $A_1, A_2$  be constant tensors and  $a$  be a constant matrix. Consider the following system:

$$1896 \begin{cases} \nabla \cdot (A_1 \nabla \chi) = 0 & \text{in } B(1)_+, \\ \nabla \cdot (A_2 \nabla \chi) + \nabla \cdot (a^T r) = 0 & \text{in } B(1)_-, \\ \nabla \cdot (a \chi) = 0 & \text{in } B(1)_-, \\ \chi|_+ = \chi|_- & \text{on } B(1) \cap \{x_d = 0\}, \\ \left. \frac{\partial \chi}{\partial \nu} \right|_+ - \left. \frac{\partial \chi}{\partial \nu} \right|_- = g_0 & \text{on } B(1) \cap \{x_d = 0\}. \end{cases}$$

1900  
1901  
1902  
1903  
1904 where  $B(1) = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ ,  $B(1)_+ = B(1) \cap \mathbb{R}_+^d$ ,  $B(1)_- = B(1) \cap \mathbb{R}_-^d$ , and  $\mathbb{R}_\pm^d = \{x \in \mathbb{R}^d : \pm x_d > 0\}$ .

1905  
1906 Let  $\chi, r$  be weak solutions of the above system. Then for all  $k \geq 0$  and  $\alpha \in [0, 1]$ , we have

$$1907 \sum_{\pm} \|\chi\|_{H^k(B(\frac{1}{2}, \pm))} \leq C (\|\chi\|_{L^2(B(1))} + |g_0|), \quad (54)$$

1908  
1909  
1910 where  $C$  is a constant independent of  $\chi, r$ , and  $B(\frac{1}{2}) = \{x \in \mathbb{R}^d : \|x\| \leq \frac{1}{2}\}$ .

1911 **Lemma 5.** Let  $A_1, A_2$  be constant tensors and  $a$  be a constant matrix. Consider the system:

$$\begin{cases}
 \nabla \cdot (A_1 \nabla \chi) = 0 & \text{in } B(1)_+, \\
 \nabla \cdot (A_2 \nabla \chi) + \nabla \cdot (a^T r) = 0 & \text{in } B(1)_-, \\
 \nabla \cdot (a \chi) = 0 & \text{in } B(1)_-, \\
 \chi|_+ = \chi|_- & \text{on } B(1) \cap \{x_d = 0\}, \\
 \left. \frac{\partial \chi}{\partial \nu} \right|_+ - \left. \frac{\partial \chi}{\partial \nu} \right|_- = g_0 & \text{on } B(1) \cap \{x_d = 0\}.
 \end{cases}$$

1919 where  $\chi, r$  are weak solutions.

1920 Define

$$l(y) = (\nabla \chi)_+(0) y \mathbf{1}_{y_d \geq 0} + (\nabla \chi)_-(0) y \mathbf{1}_{y_d \leq 0} + \chi(0), \quad q(y) = r(0),$$

1921 where  $\mathbf{1}$  denotes the indicator function. By Lemma 2, we know  $\chi(0), (\nabla \chi(x))_{\pm}(0)$  are well-defined.

$$(I - e_d \otimes e_d)(\nabla \chi)_+ = (I - e_d \otimes e_d)(\nabla \chi)_- \quad \text{on } B(1) \cap \{y_d = 0\},$$

1922 where  $e_d$  is the standard basis vector in  $\mathbb{R}^d$  and  $I$  is the identity matrix. From Lemma 3, it follows that  $(l, q) \in \mathcal{L}$ .

1923 For some  $\beta \in (0, 1)$  and all  $y \in B(\frac{1}{2})$ , we have:

$$\begin{aligned}
 |\chi(y) - l(y)| &= |\chi(y) - \chi(0) - (\nabla \chi)_{\pm}(0) y| \\
 &\leq C |y|^{\beta+1} \left( [\chi]_{C^{1,\beta}(B(\frac{1}{2})_{\pm})} \right) \\
 &\leq C |y|^{\beta+1} \left( \left( \int_{B(1)} |\chi|^2 \right)^{\frac{1}{2}} + |g_0| \right).
 \end{aligned}$$

1924 For all  $y \in B(\frac{1}{2})_-$ :

$$\begin{aligned}
 |r - q| &\leq C |y|^{\beta} \left( [r]_{C^{0,\beta}(B(\frac{1}{2})_-)} \right) \\
 &\leq C |y|^{\beta} \left( \int_{B(1)_-} |r|^2 \right)^{\frac{1}{2}}.
 \end{aligned}$$

1925 Therefore, for some  $\beta \in (0, 1)$  and all  $y \in B(\frac{t}{2})$ :

$$\begin{aligned}
 |\chi(y) - l(y)| &= |\chi(y) - \chi(0) - (\nabla \chi)_{\pm}(0) y| \\
 &\leq C \left| \frac{y}{t} \right|^{\beta+1} \left( [\chi]_{C^{1,\beta}(B(\frac{t}{2})_{\pm})} \right) \\
 &\leq C \left| \frac{y}{t} \right|^{\beta+1} \left( \left( \int_{B(t)} |\chi|^2 \right)^{\frac{1}{2}} + t |g_0| \right),
 \end{aligned}$$

1926 where  $\int$  denotes the average integral.

1927 For all  $y \in B(\frac{t}{2})_-$ :

$$\begin{aligned}
 |r - q| &\leq C \left| \frac{y}{t} \right|^{\beta} \left( [r]_{C^{0,\beta}(B(\frac{t}{2})_-)} \right) \\
 &\leq C \left| \frac{y}{t} \right|^{\beta} \left( \int_{B(t)_-} |r|^2 \right)^{\frac{1}{2}}.
 \end{aligned}$$

**Lemma 6.** Let  $A_1, A_2$  be constant tensors and  $a$  be a constant matrix. Consider the system:

$$\begin{cases} \nabla \cdot (A_1 \nabla \chi) = 0 & \text{in } B(1)_+, \\ \nabla \cdot (A_2 \nabla \chi) + \nabla \cdot (a^T r) = 0 & \text{in } B(1)_-, \\ \nabla \cdot (a \chi) = 0 & \text{in } B(1)_-, \\ \chi|_+ = \chi|_- & \text{on } B(1) \cap \{x_d = 0\}, \\ \left. \frac{\partial \chi}{\partial \nu} \right|_+ - \left. \frac{\partial \chi}{\partial \nu} \right|_- = g_0 & \text{on } B(1) \cap \{x_d = 0\}. \end{cases}$$

where  $\chi, r$  are weak solutions.

Moreover, for all  $\rho \in (0, t)$ , integrating the above inequalities yields:

$$\left( \int_{B(\rho)} |\chi - l|^2 \right)^{\frac{1}{2}} + \rho |g_0 - \zeta(l, q)| \leq \left| \frac{\rho}{t} \right|^{\beta+1} \left( \left( \int_{B(t)} |\chi|^2 \right)^{\frac{1}{2}} + t |g_0| \right). \quad (55)$$

Thus, for all  $(l', q') \in \mathcal{L}$ , by the inequality above, we have:

$$\begin{aligned} & \inf_{(l, q) \in \mathcal{L}} \left\{ \left( \int_{B(\rho)} |\chi - l|^2 \right)^{\frac{1}{2}} + \rho |g_0 - \zeta(l, q)| \right\} \\ & \leq C \left| \frac{\rho}{t} \right|^{\beta+1} \inf_{(l, q) \in \mathcal{L}} \left( \left( \int_{B(t)} |\chi - l|^2 \right)^{\frac{1}{2}} + t |g_0 - \zeta(l, q)| \right). \end{aligned} \quad (56)$$

Further, for all  $(l', q') \in \mathcal{L}$ , it follows that:

$$\begin{aligned} & \inf_{(l, q) \in \mathcal{L}} \left\{ \left( \int_{B(\rho)} |\chi - l|^2 \right)^{\frac{1}{2}} + \rho |g_0 - \zeta(l, q)| \right\} \\ & \leq C \left| \frac{\rho}{t} \right|^{\beta+1} \inf_{(l, q) \in \mathcal{L}} \left( \left( \int_{B(t)} |\chi - l|^2 \right)^{\frac{1}{2}} + t |g_0 - \zeta(l, q)| \right). \end{aligned} \quad (57)$$

**Lemma 7.** Suppose  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a non-decreasing non-negative function satisfying  $\phi(\rho) \leq C \left( \frac{\rho}{t} \right)^\beta \phi(r) + B r^\alpha$ , where  $\beta > \alpha > 0, C > 0, \forall 0 < \rho < r < R, \exists C_1$ , s.t.  $\phi(\rho) \leq C_1 \left( \frac{\rho}{t} \right)^\alpha \phi(r) + B \rho^\alpha$ .

**Theorem 2.** Suppose  $A_1, A_2, a$  are  $C^\alpha$ -Holder continuous, and let  $S_t = B(t) \cap \{x_d = 0\}$  where  $B(t) = \{x \in \mathbb{R}^d : \|x\| \leq t\}$ . Let  $\chi, r$  be weak solutions to the following system:  $\forall V \in H_0^1(B(1); \mathbb{R}^d)$ ,

$$\begin{cases} (\nabla V : A_1 \nabla \chi)_{B_+} + (\nabla V : A_2 \nabla \chi)_{B_-} + (r, \nabla \cdot (a \chi))_{B_-} = 0 \\ \nabla \cdot (a \chi) = 0 \end{cases}$$

where  $B_+ = B(1) \cap \mathbb{R}_+^d, B_- = B(1) \cap \mathbb{R}_-^d$ , and  $\mathbb{R}_\pm^d = \{x \in \mathbb{R}^d : \pm x_d > 0\}$ .

Then we have:

$$\sum_{\pm} \|\chi\|_{C^{1, \alpha}(B(\frac{1}{2})_{\pm})} \leq C \|\chi\|_{L^2(B(1))}, \quad (58)$$

where  $C$  is a constant independent of  $\chi$ , and  $C^{1, \alpha}(\Omega)$  denotes the Holder space of functions with  $\alpha$ -Holder continuous first derivatives.

2009 *Equivalent Formulation: This estimate is equivalent to showing that  $\forall \rho \in (0, \frac{1}{4})$ ,*

$$2010 \inf_{(l,q) \in \mathcal{L}} \left( \int_{B(\rho)} |\chi - l|^2 \right)^{\frac{1}{2}} \leq C \rho^{1+\alpha} \sum_{\pm} \|\chi\|_{L^2(B(\frac{3}{4}))},$$

2011  
2012  
2013 here  $\mathcal{L}$  is the space of piecewise linear functions  $(l, q)$  as defined earlier;  $B(\rho)$  is a small ball with center  
2014 arbitrary on  $S_{\frac{3}{4}} = B(\frac{3}{4}) \cap \{x_d = 0\}$  and  $f$  denotes the average integral over the domain.

2015  
2016  
2017 *Disturbance.* Suppose  $w_t, s_t$  are solutions to the following equations, where  $A_1^0 = A_1(0)$ ,  $A_2^0 = A_2(0)$ ,  
2018  $a^0 = a(0)$ :  $\forall V \in H_0^1(B(t); \mathbb{R}^d)$ ,

$$2019 \begin{cases} (\nabla V : A_1^0 \nabla w_t)_{B_+} + (\nabla V : A_2^0 \nabla w_t)_{B_-} + (s_t, \nabla \cdot (a^0 V))_{B_-} = 0, \\ \nabla \cdot (a^0 w_t) = 0 \\ w_t = \chi \\ s_t = r \end{cases} \begin{array}{l} \text{in } B_-, \\ \text{on } \partial B(t), \\ \text{on } \partial B(t)_-. \end{array}$$

2020  
2021  
2022  
2023  
2024 where  $B_+ = B(t) \cap \mathbb{R}_+^d$ ,  $B_- = B(t) \cap \mathbb{R}_-^d$ , and  $B(t) = \{x \in \mathbb{R}^d : \|x\| \leq t\}$ .

2025 By the continuity result in Lemma 4, we have  $\forall \rho \in (0, t)$ :

$$2026 \left( \int_{B(\rho)} |\nabla w_t|^2 \right)^{\frac{1}{2}} + \left( \int_{B(\rho)_-} |s_t|^2 \right)^{\frac{1}{2}} \\ 2027 \leq C \left( \left( \int_{B(t)} |\nabla w_t|^2 \right)^{\frac{1}{2}} + \left( \int_{B(t)_-} |s_t|^2 \right)^{\frac{1}{2}} \right),$$

2028  
2029  
2030  
2031  
2032  
2033 where  $C$  is a constant independent of  $t, \rho$ .

2034 Moreover, from the equations, we derive:

$$2035 (\nabla \phi, A^0 \nabla (w_t - \chi))_{B(t)} = (\nabla \phi, (A - A^0) \nabla \chi)_{B(t)} \\ 2036 + (r, \nabla \cdot (a^0 \phi))_{B(t)_-} - (s_t, \nabla \cdot (a^0 \phi))_{B(t)_-},$$

2037  
2038  
2039 where  $A^0$  denotes the piecewise constant tensor  $A_1^0$  on  $B_+$  and  $A_2^0$  on  $B_-$ .

2040 Let  $\phi = w_t - \chi$ . Substituting into the above equation, we obtain:

$$2041 \Lambda^{-1} \|\nabla w_t - \nabla \chi\|_{L^2(B(t))}^2 \leq \int_{B(t)} |A - A^0| |\nabla \chi| |\nabla w_t - \nabla \chi| \\ 2042 + \int_{B(t)_-} (r - s_t) \cdot (a^0 (-\nabla \chi)) + \int_{B(t)_-} r (a - a^0) \nabla (w_t - \chi) \\ 2043 \leq \int_{B(t)} |A - A^0| |\nabla \chi| |\nabla w_t - \nabla \chi| + \int_{B(t)_-} |r - s_t| |a^0| |\nabla \chi| \\ 2044 + \int_{B(t)_-} |r| |a - a^0| |\nabla w_t - \nabla \chi|,$$

2045  
2046  
2047  
2048  
2049  
2050  
2051 where  $\Lambda > 0$  is the ellipticity constant of  $A^0$ .

2052 By Young's inequality, this implies:

$$2053 \int_{B(t)} |\nabla w_t - \nabla \chi|^2 + \int_{B(t)_-} |r - s_t|^2 \leq C \left[ t^{2\alpha} \int_{B(t)} |\nabla \chi|^2 + t^{2\alpha} \int_{B(t)_-} |r|^2 \right], \quad (60)$$

2054  
2055  
2056  
2057 where  $C$  depends on  $\Lambda, \|A\|_{C^\alpha}, \|a\|_{C^\alpha}$ , and  $\alpha$  is the Holder exponent.  $\square$

2058 *Morrey's Estimate and Bootstrap Analysis.* By the above analysis, we obtain the following result:  
 2059  
 2060

$$\begin{aligned}
 & \int_{B(\rho)} |\chi|^2 + \int_{B(\rho)_-} |r|^2 \\
 & \leq C \left( \int_{B(\rho)} |\nabla \chi - \nabla w_t|^2 + \int_{B(\rho)} |\nabla w_t|^2 + \int_{B(\rho)_-} |r - s_t|^2 + \int_{B(\rho)_-} |s_t|^2 \right) \\
 & \leq C \left( \left( \frac{\rho}{t} \right)^d \left( \int_{B(t)} |\nabla \chi|^2 + \int_{B(t)_-} |r|^2 \right) + t^{2\alpha} \left( \int_{B(t)} |\nabla \chi|^2 + \int_{B(t)_-} |r|^2 \right) \right).
 \end{aligned} \tag{61}$$

2061  
 2062  
 2063  
 2064  
 2065  
 2066  
 2067  
 2068  
 2069  
 2070  
 2071 Define  $\Psi(r) = \int_{B(r)} |\nabla \chi|^2 + \int_{B(r)_-} |r|^2$ . Then we have:  
 2072

$$\Psi(\rho) \leq C \left( \left( \frac{\rho}{t} \right)^d \Psi(t) + t^{2\alpha} \Psi(t) \right) \quad \forall 0 < \rho < t < \frac{1}{2}.$$

2073  
 2074  
 2075  
 2076  
 2077  
 2078 By Lemma 7, it follows that:  
 2079

$$\Psi(\rho) \leq C \rho^{2\alpha} \Psi \left( \frac{1}{2} \right) \quad \forall \rho \in \left( 0, \frac{1}{2} \right).$$

2080  
 2081  
 2082  
 2083  
 2084  
 2085 Thus, we derive:  
 2086

$$\Psi(\rho) \leq C \left( \left( \frac{\rho}{t} \right)^d \Psi(t) + t^{4\alpha} \Psi(t) \right).$$

2087  
 2088  
 2089  
 2090  
 2091  
 2092 By bootstrap analysis, for all  $r < d$  and  $0 < \rho < \frac{1}{2}$ , we have:  
 2093

$$\Psi(\rho) \leq C \rho^r \Psi \left( \frac{1}{2} \right).$$

2094  
 2095  
 2096  
 2097  
 2098  
 2099 The above estimate holds for any ball with center in  $B(\frac{1}{2})$  and radius  $t$ .  
 2100

2101 Finally, we conclude that  $|\nabla \chi| \in L^{2,r}(B(\frac{1}{2}))$ , which implies  $\chi$  is  $C^\beta$ -Holder continuous for all  $\beta \in$   
 2102  $(0, 1)$ .  $\square$   
 2103  
 2104  
 2105  
 2106

2107  $C^{1,\alpha}$  Continuity. Combining Lemma 6 with the above inequality and using the Poincaré inequality, we  
 2108 have:

$$\begin{aligned}
 & \inf_{(l,q) \in \mathcal{L}} \left\{ \int_{B(\rho)} |\chi - l|^2 + \rho^{d+2} |\zeta(l, q)|^2 \right\} \\
 & \leq \inf_{(l,q) \in \mathcal{L}} \left\{ \int_{B(\rho)} |w_t - l|^2 + \rho^{d+2} |\zeta(l, q)|^2 \right\} + \int_{B(\rho)} |\chi - w_t|^2 \\
 & \leq C \left( \frac{\rho}{t} \right)^{2\beta+2+d} \inf_{(l,q) \in \mathcal{L}} \left\{ \int_{B(t)} |w_t - l|^2 + t^{d+2} |\zeta(l, q)|^2 \right\} + C \int_{B(t)} |\chi - w_t|^2 \\
 & \leq C \left( \frac{\rho}{t} \right)^{2\beta+2+d} \inf_{(l,q) \in \mathcal{L}} \left\{ \int_{B(t)} |\chi - l|^2 + t^{d+2} |\zeta(l, q)|^2 \right\} \\
 & \quad + C \left( \frac{\rho}{t} \right)^{2\beta+2+d} \int_{B(t)} |\chi - w_t|^2 + C \int_{B(t)} |\chi - w_t|^2 \\
 & \leq C \left( \frac{\rho}{t} \right)^{2\beta+2+d} \inf_{(l,q) \in \mathcal{L}} \left\{ \int_{B(t)} |\chi - l|^2 + t^{d+2} |\zeta(l, q)|^2 \right\} + Ct^2 \int_{B(t)} |\nabla \chi - \nabla w_t|^2 \\
 & \leq C \left( \frac{\rho}{t} \right)^{2\beta+2+d} \inf_{(l,q) \in \mathcal{L}} \left\{ \int_{B(t)} |\chi - l|^2 + t^{d+2} |\zeta(l, q)|^2 \right\} + Ct^{2+2\alpha} \left( \int_{B(t)} |\nabla \chi|^2 + \int_{B(t)_-} |r|^2 \right).
 \end{aligned} \tag{65}$$

2129 Define

$$\Phi(r) = \inf_{\substack{l \in \mathcal{L} \\ q \in \mathcal{L}}} \left\{ \int_{B(r)} |\chi - l|^2 + r^{d+2} |\zeta(l, q)|^2 \right\}, \tag{66}$$

2130 and

$$b(t) = \int_{B(t)} |\nabla \chi|^2 + \int_{B(t)_-} |r|^2. \tag{67}$$

2131 Then we have

$$b(t) \leq Ct^r \Psi \left( \frac{1}{2} \right) = Ct^{d-\alpha} \Psi \left( \frac{1}{2} \right) \tag{68}$$

2132 where we set  $r = d - \alpha$ . □

2133 Moreover, it follows that:

$$\Phi(\rho) \leq C \left( \frac{\rho}{t} \right)^{d+2\beta+2} \Phi(t) + Ct^{d+2+\alpha} \Psi \left( \frac{1}{2} \right). \tag{69}$$

2134 For  $\beta \in (\alpha, 1)$ , by Lemma 7, we obtain:

$$\Phi(\rho) \leq C \rho^{d+\alpha+2} \left( \Phi \left( \frac{1}{2} \right) + \Psi \left( \frac{1}{2} \right) \right). \tag{70}$$

2135 This implies  $\chi \in C^{1, \frac{\alpha}{2}}(\overline{B(\frac{1}{2})_{\pm}}; \mathbb{R}^d)$  for all  $\forall \rho \in (0, \frac{1}{2})$ . By Lemma 7, we further derive:

$$\Phi(\rho) \leq C \rho^{d+2\alpha+2} \left( \Phi \left( \frac{1}{2} \right) + \Psi \left( \frac{1}{2} \right) \right), \tag{71}$$

2136 which implies  $\chi \in C^{1,\alpha}(\overline{B(\frac{1}{2})_{\pm}}; \mathbb{R}^d)$  for all  $\rho \in (0, \frac{1}{2})$ .

2156 C.7 ESTIMATION  
21572158 We define the cut-off function as follows: Let  $\eta_\varepsilon \in C_0^\infty(\Omega)$ , which satisfies:

$$2159 \begin{cases} 0 \leq \eta_\varepsilon(x) \leq 1, & \forall x \in \Omega, \\ |\nabla \eta_\varepsilon(x)| \leq \frac{C}{\varepsilon}, & \text{within the support of } \eta_\varepsilon \text{ (} C \text{ is a constant independent of } \varepsilon \text{)}, \\ \eta_\varepsilon(x) = 1, & \text{if } \text{dist}(x, \partial\Omega) \geq 4\varepsilon, \\ \eta_\varepsilon(x) = 0, & \text{if } \text{dist}(x, \partial\Omega) \leq 3\varepsilon. \end{cases}$$

2164 And we also set this convolution operator:

$$2166 S_\varepsilon(f)(x) = \rho_\varepsilon * f(x) = \int_{\mathbb{R}^d} f(x-y)\rho_\varepsilon dy, \text{ here } \rho \in C_0^\infty(B(0, \frac{1}{2})), \rho \geq 0, \text{ and } \int_{\mathbb{R}^d} \rho dx = 1$$

2167 . We let

$$2169 w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi\left(\frac{x}{\varepsilon}\right) \eta_\varepsilon S_\varepsilon^2(\nabla u_0)$$

2170 . We let  $B(y) = A + A\nabla\chi(y) + \nabla\chi(y)A + \nabla\chi A\nabla\chi - \hat{A}$  be the Flux Corrector.2171 **Lemma 8.** Suppose  $\Omega$  is a bounded Lipschitz domain in  $\mathbb{R}^d$ ,  $\Omega_t = \{x \in \Omega : \text{dist}(x, \partial\Omega) < t\}$ ,  $t > 0$ ,  
2172  $\forall \psi \in H_0^1(\Omega, \mathbb{R}^d)$ , we have

$$2174 \left| \int_{\Omega} A\nabla w_\varepsilon : \nabla \psi dx \right| \leq C \|\nabla \chi\|_{L^2(\Omega)} \{ \varepsilon \|S_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} + \|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \} \quad (72)$$

$$2176 + C \|\nabla \psi\|_{L^2(\Omega_{4\varepsilon})} \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})}.$$

2177 *Proof.*

$$2180 A\nabla w_\varepsilon = A\nabla u_\varepsilon - A\nabla u_0 - A\nabla \chi\left(\frac{x}{\varepsilon}\right) S_\varepsilon^2(\nabla u_0) - \varepsilon A\chi(y) \nabla (\eta_\varepsilon S_\varepsilon^2(\nabla u_0))$$

$$2182 = (A\nabla u_\varepsilon - \hat{A}\nabla u_0) + \left[ (\hat{A} - A)\nabla u_0 + \left(-\hat{A} + A + \nabla\chi\left(\frac{x}{\varepsilon}\right) A + \nabla\chi A\nabla\chi\right) \eta_\varepsilon S_\varepsilon^2(\nabla u_0) \right]$$

$$2184 - \left[ A + A\nabla\chi\left(\frac{x}{\varepsilon}\right) + \nabla\chi\left(\frac{x}{\varepsilon}\right) A + \nabla\chi A\nabla\chi - \hat{A} \right] \eta_\varepsilon S_\varepsilon^2(\nabla u_0) - \varepsilon A\chi\left(\frac{x}{\varepsilon}\right) \nabla (\eta_\varepsilon S_\varepsilon^2(\nabla u_0))$$

$$2186 = (A\nabla u_\varepsilon - \hat{A}\nabla u_0) + (\hat{A} - A) (\nabla u_0 - \eta_\varepsilon S_\varepsilon^2(\nabla u_0)) + \nabla\chi A \eta_\varepsilon S_\varepsilon^2(\nabla u_0) + \nabla\chi A \nabla\chi \eta_\varepsilon S_\varepsilon^2(\nabla u_0)$$

$$2188 - B\left(\frac{x}{\varepsilon}\right) \eta_\varepsilon S_\varepsilon^2(\nabla u_0) - \varepsilon A\chi\left(\frac{x}{\varepsilon}\right) \nabla (\eta_\varepsilon S_\varepsilon^2(\nabla u_0)).$$

2189 Because of the variational form:

$$2190 \int_{\Omega} A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon : \nabla \psi + \int_{\Omega} p^\varepsilon \cdot \text{div } \psi = 0, \quad \forall \psi \in C_0^\infty(\Omega; \mathbb{R}^m)$$

2191 ,  
2192 we can get:

$$2194 \int_{\Omega} A\nabla w_\varepsilon : \nabla \psi = \int_{\Omega} A\nabla u_\varepsilon : \nabla \psi - \hat{A}\nabla u_0 : \nabla \psi + \int_{D_\varepsilon} p^\varepsilon \cdot \text{div } \psi$$

$$2196 + \int_{\Omega} (\hat{A} - A) (\nabla u_0 - \eta_\varepsilon S_\varepsilon^2(\nabla u_0)) : \nabla \psi$$

$$2198 + \int_{\Omega} [\nabla\chi A \eta_\varepsilon S_\varepsilon^2(\nabla u_0) + \nabla\chi A \nabla\chi \eta_\varepsilon S_\varepsilon^2(\nabla u_0)] : \nabla \psi \quad (73)$$

$$2200 - \int_{D_\varepsilon} p^\varepsilon \cdot \text{div } \psi - \int_{\Omega} B \eta_\varepsilon S_\varepsilon^2(\nabla u_0) : \nabla \psi$$

$$2202 - \int_{\Omega} \varepsilon A\chi\left(\frac{x}{\varepsilon}\right) \nabla (\eta_\varepsilon S_\varepsilon^2(\nabla u_0)) : \nabla \psi.$$

2205

2206

The first three terms equal zero, then we can see:

2207

2208

2209

2210

2211

2212

2213

2214

2215

This is because:

2216

2217

2218

2219

2220

2221

2222

Moreover, by the regularity of  $\chi \in W^{1,\infty}$ , we have

2223

2224

2225

2226

2227

2228

2229

2230

2231

2232

2233

2234

2235

2236

2237

2238

2239

2240

And we also have

2241

2242

2243

2244

2245

2246

2247

2248

2249

2250

2251

2252

2253

$$\begin{aligned}
\int_{\Omega} (\hat{A} - A) (\nabla u_0 - \eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0)) : \nabla \psi &\leq C \int_{\Omega} (1 - \eta_{\varepsilon}) |\nabla u_0| |\nabla \psi| \, dx \\
&+ C \int_{\Omega} \eta_{\varepsilon} |\nabla u_0 - S_{\varepsilon}^2(\nabla u_0)| |\nabla \psi| \, dx \\
&\leq C \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} \|\nabla \psi\|_{L^2(\Omega_{4\varepsilon})} \\
&+ C \|\nabla u_0 - S_{\varepsilon}(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \|\nabla \psi\|_{L^2(\Omega)}.
\end{aligned} \tag{74}$$

$$\begin{aligned}
\|\nabla u_0 - S_{\varepsilon}^2(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} &\leq \|\nabla u_0 - S_{\varepsilon}(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} \\
&+ \|S_{\varepsilon}(\nabla u_0) - S_{\varepsilon}^2(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} \\
&\leq C \|\nabla u_0 - S_{\varepsilon}(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})}.
\end{aligned} \tag{75}$$

$$\begin{aligned}
&\int_{\Omega} [\nabla \chi A \eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0) + \nabla \chi A \nabla \chi \eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0)] : \nabla \psi - \int_{D_{\varepsilon}} p^{\varepsilon} \cdot \operatorname{div} \psi \\
&\leq \int_{\Omega} [(\nabla \chi A + \nabla \chi A \nabla \chi) (\eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0) - \nabla u_0)] : \nabla \psi \\
&\quad + \int_{\Omega} (\nabla \chi A + \nabla \chi A \nabla \chi) \nabla u_0 : \nabla \psi - \int_{\omega} p_0 \operatorname{div} \psi + \mathcal{O}(\varepsilon) \\
&= \int_{\Omega} [(\nabla \chi A + \nabla \chi A \nabla \chi) (\eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0) - \nabla u_0)] : \nabla \psi \\
&\quad + \int_{\omega} \delta_{j\beta} r^{i\alpha} \frac{\partial u_0^{\alpha}}{\partial x_i} \frac{\partial \psi^{\beta}}{\partial x_j} - \int_{\omega} \frac{\partial u^i}{\partial x_j} r^{ij} \frac{\partial \psi^k}{\partial x_k} + \mathcal{O}(\varepsilon) \\
&= \int_{\Omega} [(\nabla \chi A + \nabla \chi A \nabla \chi) (\eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0) - \nabla u_0)] : \nabla \psi + \mathcal{O}(\varepsilon) \\
&\leq C \|\nabla u_0 - S_{\varepsilon}(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \|\nabla \psi\|_{L^2(\Omega)} + \|\nabla u_0\|_{\Omega_{4\varepsilon}} \|\nabla \psi\|_{L^2(\Omega_{4\varepsilon})} + \mathcal{O}(\varepsilon).
\end{aligned} \tag{76}$$

$$\begin{aligned}
\left| \int_{\Omega} B \eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0) : \nabla \psi \right| &= \left| \int_{\Omega} b_{ij}^{\alpha\beta} \left( \frac{x}{\varepsilon} \right) S_{\varepsilon}^2 \left( \frac{\partial u_0^{\beta}}{\partial x_j} \right) \left( \frac{\nabla \psi^{\alpha}}{\partial x_i} \right) \eta_{\varepsilon} \right| \\
&= \left| \int_{\Omega} \varepsilon \frac{\partial}{\partial x_k} \left( \phi_{kij}^{\alpha\beta} \right) \left( \frac{x}{\varepsilon} \right) \frac{\partial \psi^{\alpha}}{\partial x_i} S_{\varepsilon}^2 \left( \frac{\partial u_0^{\beta}}{\partial x_j} \right) \eta_{\varepsilon} \right| \\
&\leq C \varepsilon \left[ \int_{\Omega} \eta_{\varepsilon} \left| \phi \left( \frac{x}{\varepsilon} \right) \right| |\nabla \psi| |S_{\varepsilon}^2(\nabla^2 u_0)| \, dx \right. \\
&\quad \left. + \int_{\Omega} |\nabla \eta_{\varepsilon}| \left| \phi \left( \frac{x}{\varepsilon} \right) \right| |\nabla \psi| |S_{\varepsilon}^2(\nabla u_0)| \, dx \right] \\
&\leq C (\varepsilon \|\nabla \psi\|_{L^2(\Omega)} \|S_{\varepsilon}(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \\
&\quad + \|\nabla \psi\|_{L^2(\Omega_{4\varepsilon})} \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})}).
\end{aligned} \tag{77}$$

2254 By the calculations above, we can finally get the following result

$$\begin{aligned}
2255 \int_{\Omega} A\left(\frac{x}{\varepsilon}\right) \nabla u_{\varepsilon} : \nabla \psi &\leq C \|\nabla \psi_{\varepsilon}\|_{L^2(\Omega)} \left\{ \varepsilon \|S_{\varepsilon}(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} \right. \\
2256 &+ \|\nabla u_0 - S_{\varepsilon}(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \left. \right\} \\
2257 &+ C \|\nabla w_{\varepsilon}\|_{L^2(\Omega_{4\varepsilon})} \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})}.
\end{aligned} \tag{78}$$

2261  $\square$

2262 **Theorem 3.** Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain. Then for all  $0 < \varepsilon < 1$ ,

$$2263 \|\nabla w_{\varepsilon}\|_{L^2(\Omega)} \leq C \left\{ \varepsilon \|\nabla^2 u_0\|_{L^2(\Omega \setminus \Omega_{\varepsilon})} + \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} \right\}. \tag{79}$$

2264 Thus,

$$2265 \|w_{\varepsilon}\|_{H_0^1(\Omega)} \leq C \varepsilon^{\frac{1}{2}} \|u_0\|_{H^2(\Omega)}, \tag{80}$$

2266 where  $C$  is a constant depending on  $\mu$  and  $\Omega$ .

2267 *Proof.* Let  $\tilde{\eta}_{\varepsilon} \in C_0^{\infty}(\Omega)$  satisfy  $0 \leq \tilde{\eta}_{\varepsilon} \leq 1$ ,  $\tilde{\eta}_{\varepsilon} = 0$  in  $\Omega_{\varepsilon}$ ,  $\tilde{\eta}_{\varepsilon} = 1$  in  $\Omega \setminus \Omega_{\frac{3\varepsilon}{2}}$ , and  $|\nabla \tilde{\eta}_{\varepsilon}| \leq \frac{C}{\varepsilon}$ .

2270 We can obtain

$$\begin{aligned}
2271 \|\nabla u_0 - S_{\varepsilon}(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} &\leq \|\tilde{\eta}_{\varepsilon}(\nabla u_0) - S_{\varepsilon}(\tilde{\eta}_{\varepsilon} \nabla u_0)\|_{L^2(\mathbb{R}^d)} \\
2272 &\leq C \varepsilon \|\nabla(\tilde{\eta}_{\varepsilon} \nabla u_0)\|_{L^2(\mathbb{R}^d)} \\
2273 &\leq C \left\{ \varepsilon \|\nabla^2 u_0\|_{L^2(\Omega \setminus \Omega_{\varepsilon})} + \|\nabla u_0\|_{L^2(\Omega_{2\varepsilon})} \right\}.
\end{aligned} \tag{81}$$

2274 By the inequality  $\|f\|_{L^2(\Omega_{\varepsilon})} \leq C t^{\frac{1}{2}} \|f\|_{H^1(\Omega)}$ , we have

$$2275 \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} \leq C \varepsilon^{\frac{1}{2}} \|u_0\|_{H^2(\Omega)}. \tag{82}$$

2276 By Lemma 8, set  $\psi = w_{\varepsilon} \in H_0^1(\Omega; \mathbb{R}^d)$ . Using the ellipticity of  $A$ , we get

$$\begin{aligned}
2277 \|\nabla w_{\varepsilon}\|_{L^2(\Omega)} &\leq C \left\{ \varepsilon \|S_{\varepsilon}(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} + \|\nabla u_0 - S_{\varepsilon}(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \right\} \\
2278 &+ C \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} \leq C \varepsilon^{\frac{1}{2}} \|u_0\|_{H^2(\Omega)}.
\end{aligned} \tag{83}$$

2281  $\square$

2282 **Theorem 4.** Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain. Then for all  $0 < \varepsilon < 1$ , if  $u_0 \in H^2(\Omega; \mathbb{R}^d)$ , we have

$$2283 \|u_{\varepsilon} - u_0 - \varepsilon \chi\left(\frac{x}{\varepsilon}\right) \nabla u_0\|_{H^1(\Omega)} \leq C(\mu, \Omega, \|\chi\|_{\infty}) \varepsilon^{\frac{1}{2}} \|u_0\|_{W^{2,d}(\Omega)}. \tag{84}$$

2284 *Proof.* The key step is to prove:

$$2285 \left\| \varepsilon \chi\left(\frac{x}{\varepsilon}\right) \nabla u_0 - \varepsilon \chi\left(\frac{x}{\varepsilon}\right) \eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0) \right\|_{H^1(\Omega)} \leq C \varepsilon^{\frac{1}{2}} \|u_0\|_{H^2(\Omega)}. \tag{85}$$

2286 We can find that

$$\begin{aligned}
2287 &\left\| \varepsilon \chi\left(\frac{x}{\varepsilon}\right) \nabla u_0 - \varepsilon \chi\left(\frac{x}{\varepsilon}\right) \eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0) \right\|_{H^1(\Omega)} \leq C \varepsilon \left\| \chi\left(\frac{x}{\varepsilon}\right) (\nabla u_0 - \eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0)) \right\|_{L^2(\Omega)} \\
2288 &+ C \left\| \nabla \chi\left(\frac{x}{\varepsilon}\right) (\nabla u_0 - \eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0)) \right\|_{L^2(\Omega)} + C \varepsilon \left\| \chi\left(\frac{x}{\varepsilon}\right) \nabla (\nabla u_0 - \eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0)) \right\|_{L^2(\Omega)} \\
2289 &\leq C \varepsilon \left\| \nabla (\nabla u_0 - \eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0)) \right\|_{L^2(\Omega)} + C \left\| \nabla u_0 - \eta_{\varepsilon} S_{\varepsilon}^2(\nabla u_0) \right\|_{L^2(\Omega)}.
\end{aligned} \tag{86}$$

2303  
 2304  
 2305  
 2306  
 2307  
 2308  
 2309  
 2310  
 2311  
 2312  
 2313  
 2314  
 2315  
 2316  
 2317  
 2318  
 2319  
 2320  
 2321  
 2322  
 2323  
 2324  
 2325  
 2326  
 2327  
 2328  
 2329  
 2330  
 2331  
 2332  
 2333  
 2334  
 2335  
 2336  
 2337  
 2338  
 2339  
 2340  
 2341  
 2342  
 2343  
 2344  
 2345  
 2346  
 2347  
 2348  
 2349  
 2350  
 2351

On the one hand, we have

$$\begin{aligned}
 \varepsilon \|\nabla (\nabla u_0 - \eta_\varepsilon S_\varepsilon^2(\nabla u_0))\|_{L^2(\Omega)} &\leq \varepsilon \|\nabla^2 u_0\|_{L^2(\Omega)} + \varepsilon \|\nabla (\eta_\varepsilon S_\varepsilon^2(\nabla u_0))\|_{L^2(\Omega)} \\
 &\leq C\varepsilon \|u_0\|_{H^2(\Omega)} + C \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} \\
 &\leq C\varepsilon^{\frac{1}{2}} \|u_0\|_{H^2(\Omega)}.
 \end{aligned} \tag{87}$$

On the other hand,

$$\begin{aligned}
 \|\nabla u_0 - \eta_\varepsilon S_\varepsilon^2(\nabla u_0)\|_{L^2(\Omega)} &\leq C \|\nabla u_0\|_{L^2(\Omega_{5\varepsilon})} + C \|\nabla u_0 - S_\varepsilon^2(\nabla u_0)\|_{L^2(\Omega \setminus \Omega_{4\varepsilon})} \\
 &\leq C\varepsilon^{\frac{1}{2}} \|u_0\|_{H^2(\Omega)}.
 \end{aligned} \tag{88}$$

□

## C.8 SUMMARY

These conclusions is consistent to the expected conclusion (Eq. 3), since we have:

$$\begin{aligned}
 \|u_\varepsilon - u_{\text{lim}}\|_{H^1(\Omega)} &= \|u_\varepsilon - u_0 - \varepsilon \chi\left(\frac{x}{\varepsilon}\right) \nabla u_0\|_{H^1(\Omega)} \\
 &\leq C \|w_\varepsilon\|_{H^1(\Omega)} + C \|\varepsilon \chi\left(\frac{x}{\varepsilon}\right) \nabla u_0 - \varepsilon \chi\left(\frac{x}{\varepsilon}\right) \eta_\varepsilon S_\varepsilon^2(\nabla u_0)\|_{H^1(\Omega)} \\
 &\leq C\varepsilon^{\frac{1}{2}} \|u_0\|_{H^2(\Omega)}.
 \end{aligned} \tag{89}$$